From equations (9) and (11), we know that the weight functions for the eigenfunctions $X_n = \cos(\alpha_n \ln x)$ and $X_n = \cos \alpha_n s$ are $1/x$ and 1, respectively. The value of the norm $\|X_n\|$ is, however, the same regardless of whether we think of X_n as a function of x or s. This is because, with the substitution $x = \exp s$ $(s = \ln x)$, the values of the two integrals

$$\int_1^b \frac{1}{x} \cos^2(\alpha_n \ln x) dx, \qquad \int_0^{\ln b} \cos^2 \alpha_n s \, ds$$

are found to be the same. So, if we replace c by $\ln b$ and h by hb in expression (7), we see that for this problem,

$$\|X_n\|^2 = \frac{hb \ln b + \sin^2(\alpha_n \ln b)}{2hb}.$$

The normalized eigenfunctions are, therefore,

$$\phi_n(x) = \sqrt{\frac{2hb}{hb \ln b + \sin^2(\alpha_n \ln b)}} \cos(\alpha_n \ln x) \qquad (n = 1, 2, \ldots). \tag{14}$$

PROBLEMS

In Problems 1 through 5, solve directly (without referring to any other problems) for the eigenvalues and normalized eigenfunctions.

1. $X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X'(1) = 0.$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{2} \sin \alpha_n x \quad (n = 1, 2, \ldots); \quad \alpha_n = \dfrac{(2n-1)\pi}{2}.$

2. $X'' + \lambda X = 0, \qquad X(0) = 0, \qquad hX(1) + X'(1) = 0 \quad (h > 0).$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\dfrac{2h}{h + \cos^2 \alpha_n}} \sin \alpha_n x \quad (n = 1, 2, \ldots);$
$\tan \alpha_n = -\dfrac{\alpha_n}{h} \quad (\alpha_n > 0).$

3. $X'' + \lambda X = 0, \qquad X'(0) = 0, \qquad X(c) = 0.$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\dfrac{2}{c}} \cos \alpha_n x \quad (n = 1, 2, \ldots); \quad \alpha_n = \dfrac{(2n-1)\pi}{2c}.$

4. $X'' + \lambda X = 0, \qquad X(0) = 0, \qquad X(1) - X'(1) = 0.$

Suggestion: The trigonometric identity

$$\cos^2 A = \frac{1}{1 + \tan^2 A}$$

is useful in putting $\|X_n\|^2$ in a form that leads to the expression for $\phi_n(x)$ in the answer below.

Answer: $\lambda_0 = 0, \quad \lambda_n = \alpha_n^2, \quad \phi_0(x) = \sqrt{3}x, \quad \phi_n(x) = \dfrac{\sqrt{2\left(\alpha_n^2 + 1\right)}}{\alpha_n} \sin \alpha_n x$
$(n = 1, 2, \ldots)$; $\tan \alpha_n = \alpha_n$ $(\alpha_n > 0)$.

5. $X'' + \lambda X = 0, \qquad hX(0) - X'(0) = 0 \quad (h > 0), \qquad X(1) = 0.$

Answer: $\lambda_n = \alpha_n^2, \quad \phi_n(x) = \sqrt{\dfrac{2h}{h + \cos^2 \alpha_n}} \sin \alpha_n (1 - x) \quad (n = 1, 2, \ldots);$
$\tan \alpha_n = -\dfrac{\alpha_n}{h} \quad (\alpha_n > 0).$