Proof: We have

$$f - \sum c_n\phi_n = \left(f - \sum\langle f, \phi_n\rangle\phi_n\right) + \sum\left(\langle f, \phi_n\rangle - c_n\right)\phi_n.$$

Now, $f - \sum\langle f, \phi_n\rangle\phi_n$ is easily seen to be orthogonal to all ϕ_n; see the first part of the proof of Theorem 3.4. Hence, by the Pythagorean theorem (and a simple limiting argument, if there are infinitely many ϕ_n),

$$\left\|f - \sum c_n\phi_n\right\|^2 = \left\|f - \sum\langle f, \phi_n\rangle\phi_n\right\|^2 + \sum\left|\langle f, \phi_n\rangle - c_n\right|^2.$$

The last sum on the right is clearly nonnegative, and it is zero precisely when $c_n = \langle f, \phi_n\rangle$ for all n; this establishes the theorem. ∎

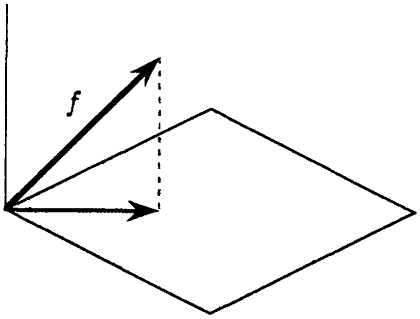

FIGURE 3.4. A vector f and its orthogonal projection onto a plane.

The pictorial intuition behind Theorem 3.8 is shown in Figure 3.4. The horizontal plane represents the space of functions (or vectors) of the form $\sum c_n\phi_n$; the sum $\sum\langle f, \phi_n\rangle\phi_n$ is the closest point to f in this plane, namely, the orthogonal projection of f onto the plane.

One situation in which Theorem 3.8 is particularly useful is when $\{\phi_n\}$ is simply a finite subset of an orthonormal basis.

Corollary 3.1. *Suppose $\{\phi_n\}_1^\infty$ is an orthonormal basis for $L^2(D)$. If $f \in L^2(D)$, the partial sum $\sum_1^N\langle f, \phi_n\rangle\phi_n$ of the series $\sum_1^\infty\langle f, \phi_n\rangle\phi_n$ is the best approximation in norm to f among all linear combinations of $\phi_1, \dots, \phi_N$.*

EXERCISES

1. Show that $\left\{e^{2\pi i(mx+ny)}\right\}_{m,n=-\infty}^{\infty}$ is an orthonormal set in $L^2(D)$ where D is any square whose sides have length one and are parallel to the coordinate axes.
2. Find constants a, b, A, B, C such that $f_0(x) = 1$, $f_1(x) = ax + b$, and $f_2(x) = Ax^2 + Bx + C$ are an orthonormal set in $L_w^2(0, \infty)$ where $w(x) = e^{-x}$. (Hint: $\int_0^\infty x^n e^{-x}dx = n!$.)

3. Let D be the unit disc $\{x^2+y^2 \leq 1\}$, and let $f_n(x,y)=(x+iy)^n$. Show that $\{f_n\}_0^\infty$ is an orthogonal set in $L^2(D)$, and compute $\|f_n\|$ for all n. (Hint: In polar coordinates, $x+iy=re^{i\theta}$ and $dx\,dy = r\,dr\,d\theta$.)
4. Suppose $\{\phi_n\}$ is an orthonormal set in $L^2_w(D)$. Show that $\{w^{1/2}\phi_n\}$ is an orthonormal set in $L^2(D)$ (with respect to the weight function 1).
5. Suppose $f:[a,b]\to[c,d]$ and $f'(x)>0$ for $x\in[a,b]$. Show that if $\{\phi_n\}$ is an orthonormal basis for $L^2(c,d)$, then $\{\phi_n\circ f\}$ is an orthonormal basis for $L^2_w(a,b)$ where $w=f'$.
6. Find an example of a sequence $\{f_n\}$ in $L^2(0,\infty)$ such that $f_n\to 0$ uniformly but $f_n \not\to 0$ in norm.
7. What is the best approximation in norm to the function $f(x)=x$ on the interval $[0,\pi]$ among all functions of the form (a) $a_0+a_1\cos x+a_2\cos 2x$, (b) $b_1\sin x+b_2\sin 2x$, (c) $a\cos x+b\sin x$?

3.5 Regular Sturm-Liouville problems

In §1.3 we arrived at the orthogonal bases $\{\cos nx\}_0^\infty$ and $\{\sin nx\}_1^\infty$ for $L^2(0,\pi)$ by solving the boundary value problems

$$u''(x)+\lambda^2u(x)=0, \qquad u'(0)=u'(\pi)=0$$

and

$$u''(x)+\lambda^2u(x)=0, \qquad u(0)=u(\pi)=0.$$

We derived the orthogonal basis $\{e^{inx}\}_{-\infty}^\infty$ for $L^2(-\pi,\pi)$ by considering periodic functions, but we could also have found it by solving the boundary value problem

$$u''(x)+\lambda^2u(x)=0, \qquad u(-\pi)=u(\pi), \quad u'(-\pi)=u'(\pi).$$

In fact, there is a large class of boundary value problems on an interval $[a,b]$ that lead to orthogonal bases for $L^2(a,b)$. These problems are the subject of the present section.

First, a bit of conceptual background from finite-dimensional linear algebra. We recall that a linear transformation $T:\mathbf{C}^k\to\mathbf{C}^k$ is called *self-adjoint* or *Hermitian* if

$$\langle T\mathbf{a},\mathbf{b}\rangle = \langle \mathbf{a}, T\mathbf{b}\rangle \quad \text{for all } \mathbf{a},\mathbf{b}\in\mathbf{C}^k.$$

(When T is described by a matrix (T_{ij}), this means that $T_{ji}=\overline{T_{ij}}$.) It is one of the basic results of linear algebra, known as the *spectral theorem* or the *principal axis theorem,* that whenever T is self-adjoint there is an orthonormal basis of $\mathbf{C}^k$ consisting of eigenvectors for T. What we are aiming for is an analogue of this theorem for differential operators acting on the space $L^2(a,b)$.

Suppose then that S and T are linear operators that are defined on certain subspaces $\mathscr{D}_S$ and $\mathscr{D}_T$ of $L^2(a,b)$ and map them into $L^2(a,b)$. We say that S and T are **adjoint** to each other (or that T is the adjoint of S, or vice versa) if

$$\langle S(f),g\rangle = \langle f,T(g)\rangle \quad \text{for all } f\in\mathscr{D}_S \text{ and } g\in\mathscr{D}_T.$$