The normalizing constants for the functions in Theorem 3.5 are, of course, $\sqrt{1/2\pi}$ for e^{inx}, $\sqrt{1/\pi}$ for $\cos nx$ and $\sin nx$ on $[-\pi, \pi]$ (except for $n = 0$), and $\sqrt{2/\pi}$ for $\cos nx$ and $\sin nx$ on $[0, \pi]$ (except for $n = 0$). With this in mind, one easily sees that the Parseval equation takes the form

$$\int_{-\pi}^{\pi} |f(x)|^2\,dx = 2\pi \sum_{-\infty}^{\infty} |c_n|^2 = \frac{\pi}{2}|a_0|^2 + \pi \sum_{1}^{\infty}\left(|a_n|^2 + |b_n|^2\right), \qquad f \in L^2(-\pi, \pi),$$

where a_n, b_n, and c_n are the Fourier coefficients of f as defined in §2.1, and

$$\int_0^{\pi} |f(x)|^2\,dx = \frac{\pi}{4}|a_0|^2 + \frac{\pi}{2}\sum_1^{\infty} |a_n|^2 = \frac{\pi}{2}\sum_1^{\infty} |b_n|^2, \qquad f \in L^2(0, \pi),$$

where a_n and b_n are the Fourier cosine and sine coefficients of f as defined in §2.4. For example, if we consider the Fourier sine series of $f(x) = x$ on $[0, \pi]$ as derived in §2.1, we find that

$$\frac{\pi}{2}\sum_1^{\infty} \frac{4}{n^2} = \int_0^{\pi} x^2\,dx = \frac{\pi^3}{3}, \quad \text{or} \quad \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

a result which we derived by other means in Exercise 3, §2.3.

Let us sum up our theorems about the convergence of Fourier series. If f is a periodic function, then the Fourier series of f converges to f

(i) absolutely, uniformly, and in norm, if f is continuous and piecewise smooth;
(ii) pointwise and in norm, if f is piecewise smooth;
(iii) in norm, if $f \in L^2(a, b)$.

These results are sufficient for virtually all practical purposes. However, as we indicated in §2.6, there is more to be said on the subject. Here we shall just mention one more result that is a natural generalization of the theorems in this section. If $1 \le p < \infty$, we define $L^p(a, b)$ to be the space of Lebesgue-integrable functions f on $[a, b]$ such that

$$\int_a^b |f(x)|^p\,dx < \infty.$$

If $p > 1$, the Fourier series of any $f \in L^p(-\pi, \pi)$ converges to f in the "L^p norm," that is, if $\{c_n\}$ are the Fourier coefficients of f,

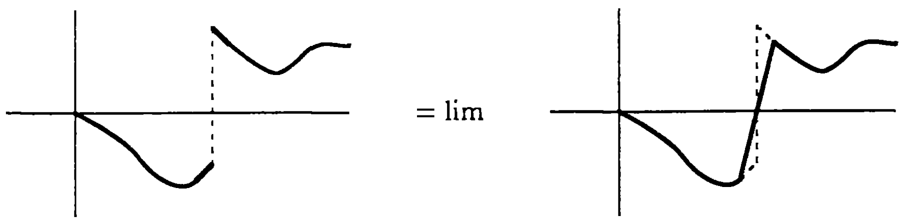
$$\int_a^b \left|\sum_{-N}^{N} c_n e^{inx} - f(x)\right|^p dx \to 0 \quad \text{as } N \to \infty.$$

However, this result is false for $p = 1$.

EXERCISES

1. Show that if $f_n \in L^2(a, b)$ and $f_n \to f$ in norm, then $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2(a, b)$. (Hint: Apply the Cauchy-Schwarz inequality to $\langle f_n - f,\ g\rangle$.)
2. Show that $\big|\,\|f\| - \|g\|\,\big| \le \|f - g\|$. (Use the triangle inequality; consider the cases $\|f\| \ge \|g\|$ and $\|f\| \le \|g\|$ separately.) Deduce that if $f_n \to f$ in norm then $\|f_n\| \to \|f\|$.

3. Show directly that any $f \in PC(a,b)$ is the limit in norm of a sequence of continuous functions on $[a,b]$, by the argument suggested by the following picture.



4. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Suppose $c > 0$ and $d \in \mathbf{R}$, and let $\psi_n(x) = c^{1/2}\phi_n(cx+d)$. Show that $\{\psi_n\}$ is an orthonormal basis for $L^2(\frac{a-d}{c}, \frac{b-d}{c})$.
5. Finish the proof of Theorem 3.5. That is, from the completeness of $\{e^{inx}\}$ on $[-\pi,\pi]$, deduce the completeness of $\{\cos nx\} \cup \{\sin nx\}$ on $[-\pi,\pi]$ and the completeness of $\{\cos nx\}$ and $\{\sin nx\}$ on $[0,\pi]$.
6. Let $\phi_n(x) = (2/l)^{1/2}\sin(n-\frac{1}{2})(\pi x/l)$. In Exercise 1, §3.2, it was shown that $\{\phi_n\}_1^\infty$ is an orthonormal set in $L^2(0,l)$. Prove that it is actually a basis, via the following argument.
   a. Let $\psi_k(x) = l^{-1/2}\sin(k\pi x/2l)$. Show that $\{\psi_k\}_1^\infty$ is an orthonormal basis for $L^2(0,2l)$. (This follows from Theorem 3.5 and Exercise 4.)
   b. If $f \in L^2(0,l)$, extend f to $[0,2l]$ by making it symmetric about the line $x = l$, that is, define the extension $\tilde{f}$ by $\tilde{f}(x) = \tilde{f}(2l-x) = f(x)$ for $x \in [0,l]$. Show that $\langle \tilde{f}, \psi_{2n}\rangle = 0$ and $\langle \tilde{f}, \psi_{2n-1}\rangle = 2^{1/2}\langle f, \phi_n\rangle$.
   c. Conclude that if $\langle f, \phi_n\rangle = 0$ for all n, then $f = 0$.
7. Show that $\left\{(2/l)^{1/2}\cos(n-\frac{1}{2})(\pi x/l)\right\}_1^\infty$ is an orthonormal basis for $L^2(0,l)$. (The argument is similar to that in Exercise 6, but this time you should extend f to be skew-symmetric about $x = l$, that is, $\tilde{f}(2l-x) = -\tilde{f}(x) = -f(x)$ for $x \in [0,l]$.)
8. Find the expansions of the functions $f(x) = 1$ and $g(x) = x$ on $[0,l]$ with respect to the orthonormal bases in Exercises 6 and 7.
9. Suppose $\{\phi_n\}$ is an orthonormal basis for $L^2(a,b)$. Show that for any $f, g \in L^2(a,b)$,
$$\langle f, g\rangle = \sum \langle f, \phi_n\rangle \overline{\langle g, \phi_n\rangle}.$$
(Note that the case $f = g$ is Parseval's equation.)
10. Evaluate the following series by applying Parseval's equation to certain of the Fourier expansions in Table 1 of §2.1.

    a. $\displaystyle\sum_1^\infty \frac{1}{n^4}$ b. $\displaystyle\sum_1^\infty \frac{1}{(2n-1)^6}$ c. $\displaystyle\sum_1^\infty \frac{n^2}{(n^2+1)^2}$

    d. $\displaystyle\sum_1^\infty \frac{\sin^2 na}{n^4}$ $(0 < a < \pi)$