

That is, if we define

$$\psi_0(x) = (1/\pi)^{1/2}, \quad \psi_n(x) = (2/\pi)^{1/2} \cos nx \quad \text{for } n > 0,$$

then $\{\psi_n\}_0^\infty$ is an orthonormal set in $PC(0, \pi)$. Moreover, if the Fourier cosine coefficients a_n of $f \in PC(0, \pi)$ are defined as before,

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx \, dx = \begin{cases} 2(1/\pi)^{1/2} \langle f, \psi_0 \rangle & \text{for } n = 0, \\ (2/\pi)^{1/2} \langle f, \psi_n \rangle & \text{for } n > 0, \end{cases}$$

we have

$$\frac{1}{2}a_0 + \sum_0^\infty a_n \cos nx = \sum_0^\infty \langle f, \psi_n \rangle \psi_n(x).$$

The reader may verify that the trigonometric form of the Fourier series on $[-\pi, \pi]$ and the Fourier sine series on $[0, \pi]$ are also instances of expansions with respect to orthonormal sets.

Now, we have been a bit cavalier in this discussion. The reader will recall that we proved the validity of Fourier expansions only for piecewise smooth functions; for functions that are merely piecewise continuous there is no guarantee that the Fourier series will converge at any given point. What this means is that we need to take a closer look at questions of convergence in the context of the ideas from vector geometry that we are now using.

EXERCISES

1. Show that $\{(2/l)^{1/2} \sin(n - \frac{1}{2})(\pi x/l)\}_1^\infty$ is an orthonormal set in $PC(0, l)$.
2. Show that $\{(2/l)^{1/2} \cos(n - \frac{1}{2})(\pi x/l)\}_1^\infty$ is an orthonormal set in $PC(0, l)$.
3. Show that $f_0(x) = 1$ and $f_1(x) = x$ are orthogonal on $[-1, 1]$, and find constants a and b so that $f_2(x) = x^2 + ax + b$ is orthogonal to both f_0 and f_1 on $[-1, 1]$. What are the normalizations of f_0 , f_1 , and f_2 ?
4. Suppose $\{\phi_n\}$ is an orthonormal set in $PC(0, l)$, and let ϕ_n^+ and ϕ_n^- be the even and odd extensions of ϕ_n to $[-l, l]$. Show that $\{2^{-1/2}\phi_n^+\} \cup \{2^{-1/2}\phi_n^-\}$ is an orthonormal set in $PC(-l, l)$. (Hint: First show that $\{2^{-1/2}\phi_n^+\}$ and $\{2^{-1/2}\phi_n^-\}$ are orthonormal, and then that $\langle \phi_n^+, \phi_m^- \rangle = 0$ for all m, n .)
5. Let $\{\phi_n : n \geq 0\}$ be an orthonormal set in $PC(-l, l)$ such that ϕ_n is even when n is even and ϕ_n is odd when n is odd. Show that $\{\sqrt{2}\phi_n : n \text{ even}\}$ and $\{\sqrt{2}\phi_n : n \text{ odd}\}$ are orthonormal sets in $PC(0, l)$.
6. Suppose $f \in PC(a, b)$ and $f(x) = \frac{1}{2}[f(x-) + f(x+)]$ for all $x \in (a, b)$. Show that if $f(x_0) \neq 0$ for some $x_0 \in (a, b)$, then $f(x) \neq 0$ for all x in some interval containing x_0 . (x_0 may be an endpoint of the interval.)
7. Show that if $f \in PC(a, b)$, $f \geq 0$, and $\int_a^b f(x) \, dx = 0$, then $f(x) = 0$ except perhaps at finitely many points. (Hint: By redefining f at its discontinuities, you can make f satisfy the conditions of Exercise 6.)