Indeed, suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set in $\mathbf{C}^k$. If a vector $\mathbf{a} \in \mathbf{C}^k$ is expressed as a linear combination of the $\mathbf{u}_j$'s,

$$\mathbf{a} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k,$$

by taking the inner product of both sides with $\mathbf{u}_j$ and using (3.10) we find that the coefficients c_j are given by

$$c_j = \langle \mathbf{a}, \mathbf{u}_j \rangle \qquad (1 \le j \le k). \tag{3.12}$$

Conversely, if $\mathbf{a}$ is any vector in $\mathbf{C}^n$, we may define the constants $c_1, \ldots, c_k$ by (3.12) and form the linear combination

$$\tilde{\mathbf{a}} = c_1\mathbf{u}_1 + \cdots + c_k\mathbf{u}_k.$$

Then the difference $\mathbf{b} = \mathbf{a} - \tilde{\mathbf{a}}$ is orthogonal to all the u_j's:

$$\langle \mathbf{b}, \mathbf{u}_j \rangle = \langle \mathbf{a}, \mathbf{u}_j \rangle - \langle \tilde{\mathbf{a}}, \mathbf{u}_j \rangle = c_j - c_j = 0.$$

But this means that $\mathbf{b} = \mathbf{0}$, for otherwise $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{b}\}$ would be an orthogonal set with $k+1$ elements, which is impossible. In other words, $\tilde{\mathbf{a}} = \mathbf{a}$, and we have the following result.

Theorem 3.1. *Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be an orthonormal set of k vectors in $\mathbf{C}^k$. For any $\mathbf{a} \in \mathbf{C}^k$ we have*

$$\mathbf{a} = \langle \mathbf{a}, \mathbf{u}_1 \rangle \mathbf{u}_1 + \cdots + \langle \mathbf{a}, \mathbf{u}_k \rangle \mathbf{u}_k.$$

Moreover,

$$\|\mathbf{a}\|^2 = |\langle \mathbf{a}, \mathbf{u}_1 \rangle|^2 + \cdots + |\langle \mathbf{a}, \mathbf{u}_k \rangle|^2.$$

Proof: The first assertion has just been proved, and the second one follows from it by the Pythagorean theorem. ∎

EXERCISES

1. Show that $\|\mathbf{a}+\mathbf{b}\|^2 + \|\mathbf{a}-\mathbf{b}\|^2 = 2\left(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2\right)$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
2. Suppose $\{\mathbf{y}_1, \ldots, \mathbf{y}_k\}$ is an orthogonal set in $\mathbf{C}^k$, not necessarily normalized. Use Theorem 3.1 to show that for any $\mathbf{a} \in \mathbf{C}^k$,

$$\mathbf{a} = \frac{\langle \mathbf{a}, \mathbf{y}_1 \rangle \mathbf{y}_1}{\|\mathbf{y}_1\|^2} + \cdots + \frac{\langle \mathbf{a}, \mathbf{y}_k \rangle \mathbf{y}_k}{\|\mathbf{y}_k\|^2}.$$

3. Let $\mathbf{y}_1 = (2, 3i, 5)$ and $\mathbf{y}_2 = (3i, 2, 0)$.
   a. Show that $\langle \mathbf{y}_1, \mathbf{y}_2 \rangle = 0$ and find a nonzero $\mathbf{y}_3$ that is orthogonal to both $\mathbf{y}_1$ and $\mathbf{y}_2$.
   b. What are the norms of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$?
   c. Use Theorem 3.1 or Exercise 2 to express the vectors $(1, 2, 3i)$ and $(0, 1, 0)$ as linear combinations of $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$.

4. Let $\mathbf{u}_1 = \frac{1}{3}(1, 2i, -2i, 0)$, $\mathbf{u}_2 = \frac{1}{5}(2-4i, -2, i, 0)$, $\mathbf{u}_3 = \frac{1}{15}(4+2i, 5+8i, 4+10i, 0)$, and $\mathbf{u}_4 = (0, 0, 0, i)$.
   a. Show that $\{\mathbf{u}_1, \dots, \mathbf{u}_4\}$ is an orthonormal set in $\mathbf{C}^4$.
   b. Express the vectors $(1, 0, 0, 0)$ and $(2, 10-i, 10-9i, -3)$ as linear combinations of $\mathbf{u}_1, \dots, \mathbf{u}_4$ by using Theorem 3.1.
5. Suppose $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ is an orthonormal set in $\mathbf{C}^k$ with $m < k$. Show that for any $\mathbf{a} \in \mathbf{C}^k$ there is a unique set of constants $\{c_1, \dots, c_m\}$ such that $\mathbf{a} - \sum_1^m c_j \mathbf{u}_j$ is orthogonal to all the $\mathbf{u}_j$'s, and determine these constants explicitly. (Hint: Consider the proof of Theorem 3.1.)

The following problems deal with $k \times k$ complex matrices $T = (T_{ij})$. We recall that if $T = (T_{ij})$ and $S = (S_{ij})$ are $k \times k$ matrices, TS is the matrix whose (ij)th component is $\sum_l T_{il} S_{lj}$, and if $\mathbf{a} \in \mathbf{C}^k$, $T\mathbf{a}$ is the vector whose ith component is $\sum_j T_{ij} a_j$. The (Hermitian) **adjoint** of the matrix T is the matrix T^* obtained by interchanging rows and columns and taking complex conjugates, that is, $(T^*)_{ij} = \overline{T_{ji}}$.

6. Show that $\langle T\mathbf{a}, \mathbf{b}\rangle = \langle \mathbf{a}, T^*\mathbf{b}\rangle$ for all $\mathbf{a}, \mathbf{b} \in \mathbf{C}^k$.
7. Show that if $T = T^*$, the "product" defined by $\langle \mathbf{a}, \mathbf{b}\rangle_T = \langle T\mathbf{a}, \mathbf{b}\rangle$ satisfies properties (3.3) and (3.4).
8. Let $\mathbf{t}_j = (T_{1j}, \dots, T_{kj})$ be the vector that makes up the jth row of T. Show that the following properties of the matrix T are equivalent. (Hint: Show that the (ij)th component of T^*T is $\langle \mathbf{t}_j, \mathbf{t}_i\rangle$.)
   (i) $\{\mathbf{t}_1, \dots, \mathbf{t}_k\}$ is an orthonormal basis for $\mathbf{C}^k$.
   (ii) T^*T is the identity matrix, i.e., $(T^*T)_{ij} = \delta_{ij}$.
   (iii) $\|T\mathbf{a}\| = \|\mathbf{a}\|$ for all $\mathbf{a} \in \mathbf{C}^k$.
9. Show that $|\langle \mathbf{a}, \mathbf{b}\rangle| = \|\mathbf{a}\|\,\|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are complex scalar multiples of one another, and that $\|\mathbf{a}+\mathbf{b}\| = \|\mathbf{a}\| + \|\mathbf{b}\|$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are positive scalar multiples of one another. (Examine the proofs of the Cauchy-Schwarz and triangle inequalities to see when equality holds.)

3.2 Functions and inner products

A vector $\mathbf{a} = (a_1, \dots, a_k)$ in $\mathbf{C}^k$ can be regarded as a function on the set $\{1, \dots, k\}$ that assigns to the integer j the jth component $\mathbf{a}(j) = a_j$, and with this notation we can write the inner product and norm as follows:

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_1^k \mathbf{a}(j)\overline{\mathbf{b}(j)}, \qquad \|\mathbf{a}\| = \left(\sum_1^k |\mathbf{a}(j)|^2\right)^{1/2}. \tag{3.13}$$

We now make a leap of imagination: Consider the space $PC(a, b)$ of piecewise continuous functions on the interval $[a, b]$, and think of functions $f \in PC(a, b)$ as infinite-dimensional vectors whose "components" are the values $f(x)$ as x ranges over the interval $[a, b]$. The operations of vector addition and scalar multiplication are just the usual addition of functions and multiplication of functions by