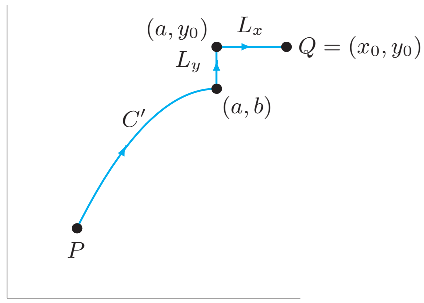


Figure 18.30: The path $C' + L_y + L_x$ is used to show $f_x = F_1$

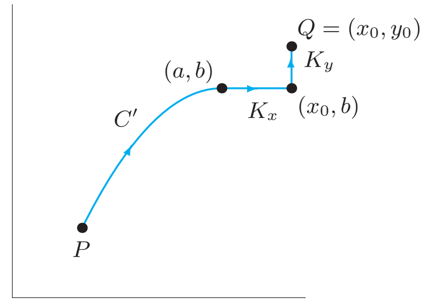


Figure 18.31: The path $C' + K_x + K_y$ is used to show $f_y = F_2$

The first two integrals do not involve x_0. Thinking of x_0 as a variable and differentiating with respect to it gives

$$f_{x_0}(x_0, y_0) = \frac{\partial}{\partial x_0}\int_{C'} \vec{F}\cdot d\vec{r} + \frac{\partial}{\partial x_0}\int_b^{y_0} F_2(a, y)dy + \frac{\partial}{\partial x_0}\int_a^{x_0} F_1(x, y_0)dx$$
$$= 0 + 0 + F_1(x_0, y_0) = F_1(x_0, y_0),$$

and thus

$$f_x(x, y) = F_1(x, y).$$

A similar calculation for y using the path from P to Q shown in Figure 18.31 gives

$$f_{y_0}(x_0, y_0) = F_2(x_0, y_0).$$

Therefore, as we claimed,

$$\text{grad}\, f = f_x\vec{i} + f_y\vec{j} = F_1\vec{i} + F_2\vec{j} = \vec{F}.$$

Exercises and Problems for Section 18.3

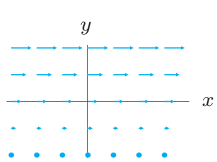
Exercises

1. If $\vec{F} = \text{grad}(x^2 + y^4)$, find $\int_C \vec{F}\cdot d\vec{r}$ where C is the quarter of the circle $x^2 + y^2 = 4$ in the first quadrant, oriented counterclockwise.

2. If $\vec{F} = \text{grad}(\sin(xy) + e^z)$, find $\int_C \vec{F}\cdot d\vec{r}$ where C consists of a line from $(0, 0, 0)$ to $(0, 0, 1)$ followed by a line to $(0, \sqrt{2}, 3)$, followed by a line to $(\sqrt{2}, \sqrt{5}, 2)$.

In Exercises 3–6, let C be the curve consisting of a square of side 2, centered at the origin with sides on the lines $x = \pm 1$, $y = \pm 1$ and traversed counterclockwise. What is the sign of the line integrals of the vector fields around the curve C? Indicate whether each vector field is path-independent.

3.



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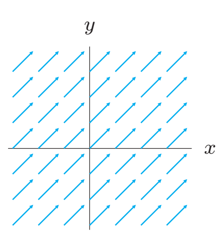

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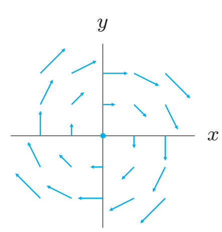

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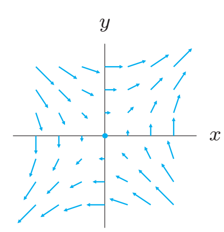
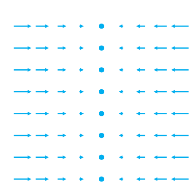


In Exercises 7–12, does the vector field appear to be path-independent (conservative)?

7.



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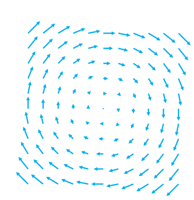

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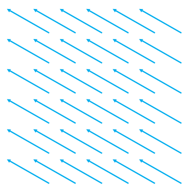

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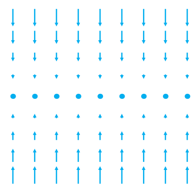

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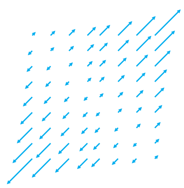

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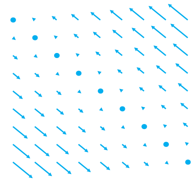

13. Find f if $\operatorname{grad} f = 2xy\vec{i} + x^2\vec{j}$.

14. Find f if $\operatorname{grad} f = 2xy\vec{i} + (x^2 + 8y^3)\vec{j}$.

15. Find f if $\operatorname{grad} f = (yze^{xyz} + z^2\cos(xz^2))\vec{i} + xze^{xyz}\vec{j} + (xye^{xyz} + 2xz\cos(xz^2))\vec{k}$.

16. Let $f(x, y, z) = x^2 + 2y^3 + 3z^4$ and $\vec{F} = \operatorname{grad} f$. Find $\int_C \vec{F} \cdot d\vec{r}$ where C consists of four line segments from $(4, 0, 0)$ to $(4, 3, 0)$ to $(0, 3, 0)$ to $(0, 3, 5)$ to $(0, 0, 5)$.

In Exercises 17–25, use the Fundamental Theorem of Line Integrals to calculate $\int_C \vec{F} \cdot d\vec{r}$ exactly.

17. $\vec{F} = 3x^2\vec{i} + 4y^3\vec{j}$ around the top of the unit circle from $(1, 0)$ to $(-1, 0)$.

18. $\vec{F} = (x+2)\vec{i} + (2y+3)\vec{j}$ and C is the line from $(1, 0)$ to $(3, 1)$.

19. $\vec{F} = 2\sin(2x+y)\vec{i} + \sin(2x+y)\vec{j}$ along the path consisting of a line from $(\pi, 0)$ to $(2, 5)$ followed by a line to $(5\pi, 0)$ followed by a quarter circle to $(0, 5\pi)$.

20. $\vec{F} = 2x\vec{i} - 4y\vec{j} + (2z - 3)\vec{k}$ and C is the line from $(1, 1, 1)$ to $(2, 3, -1)$.

21. $\vec{F} = x^{2/3}\vec{i} + e^{7y}\vec{j}$, and C is the unit circle oriented clockwise.

22. $\vec{F} = x^{2/3}\vec{i} + e^{7y}\vec{j}$, and C is the quarter of the unit circle in the first quadrant, traced counterclockwise from $(1, 0)$ to $(0, 1)$.

23. $\vec{F} = ye^{xy}\vec{i} + xe^{xy}\vec{j} + (\cos z)\vec{k}$ along the curve consisting of a line from $(0, 0, \pi)$ to $(1, 1, \pi)$ followed by the parabola $z = \pi x^2$ in the plane $y = 1$ to the point $(3, 1, 9\pi)$.

24. $\vec{F} = y\sin(xy)\vec{i} + x\sin(xy)\vec{j}$ and C is the parabola $y = 2x^2$ from $(1, 2)$ to $(3, 18)$.

25. $\vec{F} = 2xy^2ze^{x^2y^2z}\vec{i} + 2x^2yze^{x^2y^2z}\vec{j} + x^2y^2e^{x^2y^2z}\vec{k}$ and C is the circle of radius 1 in the plane $z = 1$, centered on the z-axis, starting at $(1, 0, 1)$ and oriented counterclockwise viewed from above.

Problems

26. Let $\vec{v} = \operatorname{grad}(x^2 + y^2)$. Consider the path C which is a line between any two of the following points:
$(0, 0); (5, 0); (-5, 0); (0, 6); (0, -6); (5, 4); (-3, -5)$.
Suppose you want to choose the path C in order to maximize $\int_C \vec{v} \cdot d\vec{r}$. What point should be the start of C? What point should be the end of C? Explain your answer.

27. Let $\vec{F} = \operatorname{grad}(2x^2 + 3y^2)$. Which one of the three paths PQ, QR, and RS in Figure 18.32 should you choose as C in order to maximize $\int_C \vec{F} \cdot d\vec{r}$?

28. Compute $\displaystyle\int_C \left(\cos(xy)e^{\sin(xy)}(y\vec{i} + x\vec{j}) + \vec{k}\right) \cdot d\vec{r}$ where C is the line from $(\pi, 2, 5)$ to $(0.5, \pi, 7)$.

29. The vector field $\vec{F}(x, y) = x\vec{i} + y\vec{j}$ is path-independent. Compute geometrically the line integrals over the three paths A, B, and C shown in Figure 18.33 from $(1, 0)$ to $(0, 1)$ and check that they are equal. Here A is a portion of a circle, B is a line, and C consists of two line segments meeting at a right angle.

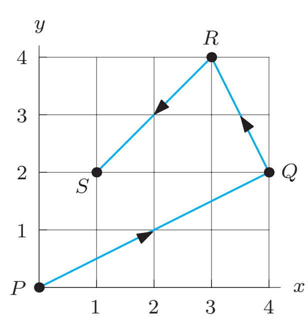

Figure 18.32

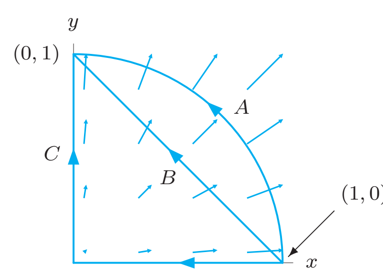

Figure 18.33