(b) Using $D(t)$, we get

$$\int_L \vec{F} \cdot d\vec{r} = \int_0^{\ln 2} \left(\left(3(e^t - 1) - (e^t - 1)\right)\vec{i} + (e^t - 1)\vec{j}\right) \cdot (e^t\vec{i} + e^t\vec{j})\, dt$$
$$= \int_0^{\ln 2} 3(e^{2t} - e^t)\, dt = 3\left(\frac{e^{2t}}{2} - e^t\right)\Bigg|_0^{\ln 2} = \frac{3}{2}.$$

The fact that both answers are the same illustrates that the value of a line integral is independent of the parameterization of the path. Problems 38–40 at the end of this section give another way of seeing this.

Exercises and Problems for Section 18.2

Exercises

In Exercises 1–3, write $\int_C \vec{F} \cdot d\vec{r}$ in the form $\int_a^b g(t)dt$. (Give a formula for g and numbers for a and b. You do not need to evaluate the integral.)

1. $\vec{F} = y\vec{i} + x\vec{j}$ and C is the semicircle from $(0,1)$ to $(0,-1)$ with $x > 0$.

2. $\vec{F} = x\vec{i} + z^2\vec{k}$ and C is the line from $(0,1,0)$ to $(2,3,2)$.

3. $\vec{F} = (\cos x)\vec{i} + (\cos y)\vec{j} + (\cos z)\vec{k}$ and C is the unit circle in the plane $z = 10$, centered on the z-axis and oriented counterclockwise when viewed from above.

In Exercises 4–8, find the line integral.

4. $\int_C (3\vec{i} + (y+5)\vec{j}) \cdot d\vec{r}$ where C is the line from $(0,0)$ to $(0,3)$.

5. $\int_C (2x\vec{i} + 3y\vec{j}) \cdot d\vec{r}$ where C is the line from $(1,0,0)$ to $(5,0,0)$.

6. $\int_C (2y^2\vec{i} + x\vec{j}) \cdot d\vec{r}$ where C is the line segment from $(3,1)$ to $(0,0)$.

7. $\int_C (x\vec{i} + y\vec{j}) \cdot d\vec{r}$ where C is the semicircle with center at $(2,0)$ and going from $(3,0)$ to $(1,0)$ in the region $y > 0$.

8. Find $\int_C ((x^2+y)\vec{i} + y^3\vec{j}) \cdot d\vec{r}$ where C consists of the three line segments from $(4,0,0)$ to $(4,3,0)$ to $(0,3,0)$ to $(0,3,5)$.

In Exercises 9–23, find $\int_C \vec{F} \cdot d\vec{r}$ for the given $\vec{F}$ and C.

9. $\vec{F} = 2\vec{i} + \vec{j}$; C is the x-axis from $x = 10$ to $x = 7$.

10. $\vec{F} = 3\vec{j} - \vec{i}$; C is the line $y = x$ from $(1,1)$ to $(5,5)$.

11. $\vec{F} = x\vec{i} + y\vec{j}$ and C is the line from $(0,0)$ to $(3,3)$.

12. $\vec{F} = y\vec{i} - x\vec{j}$ and C is the right-hand side of the unit circle, starting at $(0,1)$.

13. $\vec{F} = x^2\vec{i} + y^2\vec{j}$ and C is the line from the point $(1,2)$ to the point $(3,4)$.

14. $\vec{F} = -y\sin x\vec{i} + \cos x\vec{j}$ and C is the parabola $y = x^2$ between $(0,0)$ and $(2,4)$.

15. $\vec{F} = y^3\vec{i} + x^2\vec{j}$ and C is the line from $(0,0)$ to $(3,2)$.

16. $\vec{F} = 2y\vec{i} - (\sin y)\vec{j}$ counterclockwise around the unit circle C starting at the point $(1,0)$.

17. $\vec{F} = \ln y\vec{i} + \ln x\vec{j}$ and C is the curve given parametrically by $(2t, t^3)$, for $2 \le t \le 4$.

18. $\vec{F} = x\vec{i} + 6\vec{j} - \vec{k}$, and C is the line $x = y = z$ from $(0,0,0)$ to $(2,2,2)$.

19. $\vec{F} = (2x - y + 4)\vec{i} + (5y + 3x - 6)\vec{j}$ and C is the triangle with vertices $(0,0)$, $(3,0)$, $(3,2)$ traversed counterclockwise.

20. $\vec{F} = x\vec{i} + 2zy\vec{j} + x\vec{k}$ and C is $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ for $1 \le t \le 2$.

21. $\vec{F} = x^3\vec{i} + y^2\vec{j} + z\vec{k}$ and C is the line from the origin to the point $(2,3,4)$.

22. $\vec{F} = -y\vec{i} + x\vec{j} + 5\vec{k}$ and C is the helix $x = \cos t$, $y = \sin t$, $z = t$, for $0 \le t \le 4\pi$.

23. $\vec{F} = e^y\vec{i} + \ln(x^2+1)\vec{j} + \vec{k}$ and C is the circle of radius 2 centered at the origin in the yz-plane in Figure 18.25.

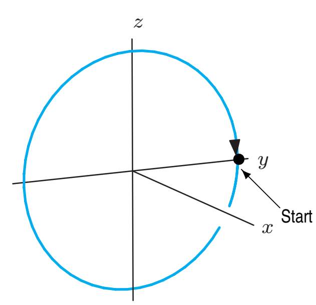

Figure 18.25

In Exercises 24–25, express the line integral $\int_C \vec{F} \cdot d\vec{r}$ in differential notation.

24. $\vec{F} = 3x\vec{i} - y\sin x\vec{j}$

25. $\vec{F} = y^2\vec{i} + z^2\vec{j} + (x^2 - 5)\vec{k}$

In Exercises 26–27, find $\vec{F}$ so that the line integral equals $\int_C \vec{F} \cdot d\vec{r}$.

26. $\int_C (x+2y)dx + x^2 y dy$

27. $\int_C e^{-3y} dx - yz(\sin x)dy + (y+z)dy$

Evaluate the line integrals in Exercises 28–29.

28. $\int_C y dx + x dy$ where C is the parameterized path $x = t^2$, $y = t^3$, $1 \le t \le 5$.

29. $\int_C dx + y dy + z dz$ where C is one turn of the helix $x = \cos t$, $y = \sin t$, $z = 3t$, $0 \le t \le 2\pi$.

Evaluate the line integrals in Exercises 30–31.

30. $\int_C 3y dx + 4x dy$ where C is the straight-line path from $(1,3)$ to $(5,9)$.

31. $\int_C x dx + z dy - y dz$ where C is the circle of radius 3 in the yz plane centered at the origin, oriented counterclockwise when viewed from the positive y-axis.

Problems

32. In Example 6 on page 971 we integrated $\vec{F} = (3x - y)\vec{i} + x\vec{j}$ over two parameterizations of the line from $(0,0)$ to $(1,1)$, getting $3/2$ each time. Now compute the line integral along two different paths with the same endpoints, and show that the answers are different.

(a) The path (t, t^2), with $0 \le t \le 1$
(b) The path (t^2, t), with $0 \le t \le 1$

33. Curves C_1 and C_2 are parametrized as follows:

$$C_1 \text{ is } (x(t), y(t)) = (0, t) \quad \text{for } -1 \le t \le 1$$
$$C_2 \text{ is } (x(t), y(t)) = (\cos t, \sin t) \quad \text{for } \frac{\pi}{2} \le t \le \frac{3\pi}{2}.$$

(a) Sketch C_1 and C_2 with arrows showing their orientation.
(b) Suppose $\vec{F} = (x+3y)\vec{i} + y\vec{j}$. Calculate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve given by $C = C_1 + C_2$.

34. Calculate the line integral of $\vec{F} = (3x-y)\vec{i} + x\vec{j}$ along the line segment L from $(0,0)$ to $(1,1)$ using each of the parameterizations

(a) $B(t) = (2t, 2t), \quad 0 \le t \le 1/2$
(b) $C(t) = \left(\frac{t^2-1}{3}, \frac{t^2-1}{3}\right), \quad 1 \le t \le 2$

35. Let $\vec{F} = -y\vec{i} + x\vec{j}$ and let C be the unit circle oriented counterclockwise.

(a) Show that $\vec{F}$ has a constant magnitude of 1 on C.
(b) Show that $\vec{F}$ is always tangent to the circle C.
(c) Show that $\int_C \vec{F} \cdot d\vec{r} =$ Length of C.

36. A spiral staircase in a building is in the shape of a helix of radius 5 meters. Between two floors of the building, the stairs make one full revolution and climb by 4 meters. A person carries a bag of groceries up two floors. The combined mass of the person and the groceries is 70 kg and the gravitational force is $70g$ downward, where g is the acceleration due to gravity. Calculate the work done by the person against gravity.

37. If C is $\vec{r} = (t+1)\vec{i} + 2t\vec{j} + 3t\vec{k}$ for $0 \le t \le 1$, we know $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 5$. Find the value of the integrals:

(a) $\int_1^0 \vec{F}((t+1)\vec{i} + 2t\vec{j} + 3t\vec{k}) \cdot (\vec{i} + 2\vec{j} + 3\vec{k})dt$
(b) $\int_0^1 \vec{F}((t^2+1)\vec{i} + 2t^2\vec{j} + 3t^2\vec{k}) \cdot (2t\vec{i} + 4t\vec{j} + 6t\vec{k})dt$
(c) $\int_{-1}^1 \vec{F}((t^2+1)\vec{i} + 2t^2\vec{j} + 3t^2\vec{k}) \cdot (2t\vec{i} + 4t\vec{j} + 6t\vec{k})dt$

In Example 6 on page 971 two parameterizations, $A(t)$, and $D(t)$, are used to convert a line integral into a definite integral. In Problem 34, two other parameterizations, $B(t)$ and $C(t)$, are used on the same line integral. In Problems 38–40 show that two definite integrals corresponding to two of the given parameterizations are equal by finding a substitution which converts one integral to the other. This gives us another way of seeing why changing the parameterization of the curve does not change the value of the line integral.

38. $A(t)$ and $B(t)$

39. $A(t)$ and $C(t)$

40. $A(t)$ and $D(t)$

41. Suppose C is the line segment from the point $(0,0)$ to the point $(4,12)$ and $\vec{F} = xy\vec{i} + x\vec{j}$.

(a) Is $\int_C \vec{F} \cdot d\vec{r}$ greater than, less than, or equal to zero? Give a geometric explanation.
(b) A parameterization of C is $(x(t), y(t)) = (t, 3t)$ for $0 \le t \le 4$. Use this to compute $\int_C \vec{F} \cdot d\vec{r}$.
(c) Suppose a particle leaves the point $(0,0)$, moves along the line toward the point $(4,12)$, stops before reaching it and backs up, stops again and reverses direction, then completes its journey to the endpoint. All travel takes place along the line segment joining the point $(0,0)$ to the point $(4,12)$. If we call this path C', explain why $\int_{C'} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r}$.
(d) A parameterization for a path like C' is given, for $0 \le t \le 4$, by

$$(x(t), y(t)) = \left(\frac{t^3 - 6t^2 + 11t}{3}, (t^3 - 6t^2 + 11t)\right).$$

Check that this parameterization begins at the point $(0,0)$ and ends at the point $(4,12)$. Check also that all points of C' lie on the line segment connecting the point $(0,0)$ to the point $(4,12)$. What are the values of t at which the particle changes direction?
(e) Find $\int_{C'} \vec{F} \cdot d\vec{r}$ using the parameterization in part (d). Do you get the same answer as in part (b)?