is given by its projection onto the z-axis, which can be seen from the figure to be $F\cos\phi$. The distance from the mass $\delta\Delta V$ to the unit mass at the origin is the spherical coordinate ρ. Therefore, the z-component of the force due to the small piece ΔV is

$$\begin{matrix} z\text{-component} \\ \text{of force} \end{matrix} = \frac{G(\delta\Delta V)(1)}{\rho^2}\cos\phi.$$

Adding the contributions of the small pieces, we get a vertical force with magnitude

$$\begin{aligned} F &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(\frac{G\delta}{\rho^2}\right)(\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} G\delta(\cos\phi\sin\phi)\rho\Big|_{\rho=0}^{\rho=a} d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/2} G\delta a\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{2\pi} G\delta a\left(-\frac{(\cos\phi)^2}{2}\right)\Big|_{\phi=0}^{\phi=\pi/2} d\theta \\ &= \int_0^{2\pi} G\delta a\left(\frac{1}{2}\right)d\theta = G\delta a\pi. \end{aligned}$$

The integral in this example is improper because the region of integration contains the origin, where the force is undefined. However, it can be shown that the result is nevertheless correct.

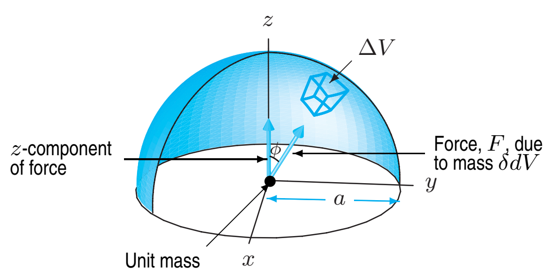


Figure 16.49: Gravitational force of hemisphere on mass at origin

Exercises and Problems for Section 16.5

Exercises

1. Match the equations in (a)–(f) with one of the surfaces in (I)–(VII).

(a) $x = 5$	**(b)** $x^2 + z^2 = 7$	**(c)** $\rho = 5$
(d) $z = 1$	**(e)** $r = 3$	**(f)** $\theta = 2\pi$

(I) Cylinder, centered on x-axis.
(II) Cylinder, centered on y-axis.
(III) Cylinder, centered on z-axis.
(IV) Plane, perpendicular to the x-axis.
(V) Plane, perpendicular to the y-axis.
(VI) Plane, perpendicular to the z-axis.
(VII) Sphere.

In Exercises 2–7, find an equation for the surface.

2. The vertical plane $y = x$ in cylindrical coordinates.

3. The top half of the sphere $x^2+y^2+z^2 = 1$ in cylindrical coordinates.

4. The cone $z = \sqrt{x^2 + y^2}$ in cylindrical coordinates.

5. The cone $z = \sqrt{x^2 + y^2}$ in spherical coordinates.

6. The plane $z = 10$ in spherical coordinates.

7. The plane $z = 4$ in spherical coordinates.

In Exercises 8–9, evaluate the triple integrals in cylindrical coordinates over the region W.

8. $f(x, y, z) = \sin(x^2 + y^2)$, W is the solid cylinder with height 4 and with base of radius 1 centered on the z axis at $z = -1$.

9. $f(x, y, z) = x^2 + y^2 + z^2$, W is the region $0 \le r \le 4$, $\pi/4 \le \theta \le 3\pi/4$, $-1 \le z \le 1$.

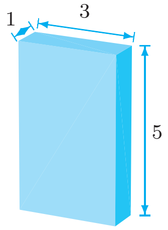
In Exercises 10–11, evaluate the triple integrals in spherical coordinates.

10. $f(\rho, \theta, \phi) = \sin\phi$, over the region $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$, $1 \le \rho \le 2$.

11. $f(x, y, z) = 1/(x^2 + y^2 + z^2)^{1/2}$ over the bottom half of the sphere of radius 5 centered at the origin.

For Exercises 12–18, choose coordinates and set up a triple integral, including limits of integration, for a density function f over the region.

12.



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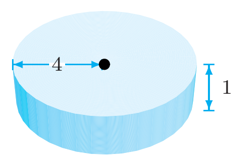

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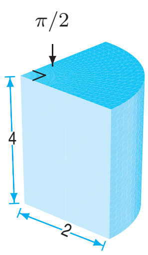

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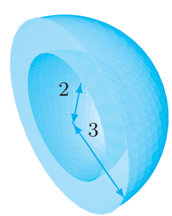
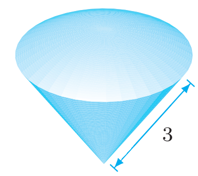
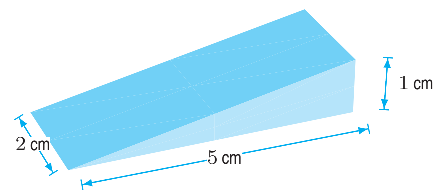


16. A piece of a sphere; angle at the center is $\pi/3$.



17.



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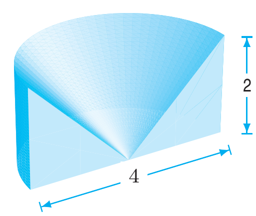


Problems

19. Write a triple integral in cylindrical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $dz\,dr\,d\theta$.

20. Write a triple integral in spherical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $d\theta\,d\rho\,d\phi$.

If W is the region in Figure 16.50, what are the limits of integration in Exercises 21–23?

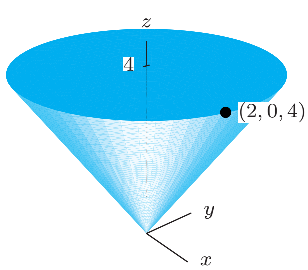


Figure 16.50: Cone with flat top, symmetric about z-axis

21. $\displaystyle\int_?^? \int_?^? \int_?^? f(r, \theta, z) r \, dz \, dr \, d\theta$

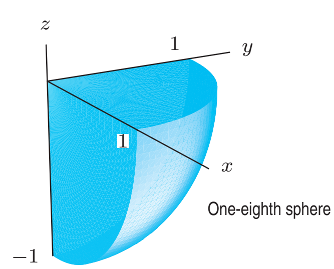
22. $\displaystyle\int_?^? \int_?^? \int_?^? g(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

23. $\displaystyle\int_?^? \int_?^? \int_?^? h(x, y, z) \, dz \, dy \, dx$

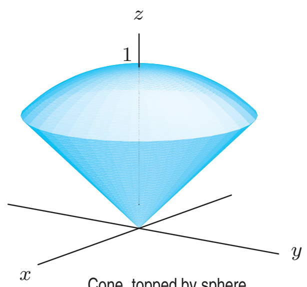
For the regions W shown in Problems 24–26, write the limits of integration for $\int_W dV$ in the following coordinates:

(a) Cartesian **(b)** Cylindrical **(c)** Spherical

24.

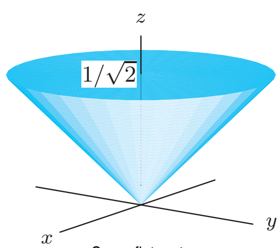


25.



Cone, topped by sphere
of radius 1 centered at origin, 90° at vertex

26.



Cone, flat on top,
$\pi/2$ at vertex

27. Write a triple integral representing the volume above the cone $z = \sqrt{x^2+y^2}$ and below the sphere of radius 2 centered at the origin. Include limits of integration but do not evaluate. Use:

(a) Cylindrical coordinates
(b) Spherical coordinates

28. Write a triple integral representing the volume of the region between spheres of radius 1 and 2, both centered at the origin. Include limits of integration but do not evaluate. Use:

(a) Spherical coordinates.
(b) Cylindrical coordinates. Write your answer as the difference of two integrals.

In Problems 29–34, write a triple integral including limits of integration that gives the specified volume.

29. Under $\rho = 3$ and above $\phi = \pi/3$.

30. Under $\rho = 3$ and above $z = r$.

31. The region between $z = 5$ and $z = 10$, with $2 \le x^2+y^2 \le 3$ and $0 \le \theta \le \pi$.

32. Between the cone $z = \sqrt{x^2+y^2}$ and the first quadrant of the xy-plane, with $x^2+y^2 \le 7$.

33. The cap of the solid sphere $x^2+y^2+z^2 \le 10$ cut off by the plane $z = 1$.

34. Below the cone $z = r$, above the xy-plane, and inside the sphere $x^2+y^2+z^2 = 8$.

35. **(a)** Write an integral (including limits of integration) representing the volume of the region inside the cone $z = \sqrt{3(x^2+y^2)}$ and below the plane $z = 1$.
(b) Evaluate the integral.

36. Find the volume between the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 10 + x$ above the disk $x^2+y^2 \le 1$.

37. Find the volume between the cone $x = \sqrt{y^2+z^2}$ and the sphere $x^2+y^2+z^2 = 4$.

38. The sphere of radius 2 centered at the origin is sliced horizontally at $z = 1$. What is the volume of the cap above the plane $z = 1$?

39. Suppose W is the region outside the cylinder $x^2+y^2 = 1$ and inside the sphere $x^2+y^2+z^2 = 2$. Calculate

$$\int_W (x^2+y^2)\,dV.$$

40. Write a triple integral representing the volume of a slice of the cylindrical cake of height 2 and radius 5 between the planes $\theta = \pi/6$ and $\theta = \pi/3$. Evaluate this integral.

41. Write a triple integral representing the volume of the cone in Figure 16.51 and evaluate it.

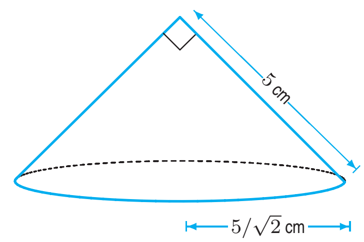


Figure 16.51

Without performing the integration, decide whether each of the integrals in Problems 42–43 is positive, negative, or zero. Give reasons for your decision.

42. W_1 is the unit ball, $x^2+y^2+z^2 \le 1$.

(a) $\int_{W_1} \sin\phi\,dV$ **(b)** $\int_{W_1} \cos\phi\,dV$

43. W_2 is $0 \le z \le \sqrt{1-x^2-y^2}$, the top half of the unit ball.

(a) $\int_{W_2} (z^2 - z)\,dV$ **(b)** $\int_{W_2} (-xz)\,dV$

44. The insulation surrounding a pipe of length l is the region between two cylinders with the same axis. The inner cylinder has radius a, the outer radius of the pipe, and the insulation has thickness h. Write a triple integral, including limits of integration, giving the volume of the insulation. Evaluate the integral.

45. Assume p, q, r are positive constants. Find the volume contained between the coordinate planes and the plane

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1.$$