(d) This is another polar region: it is a piece of a ring in which r goes from 1 to 2. Since it is in the second quadrant, θ goes from $\pi/2$ to π. The integral is

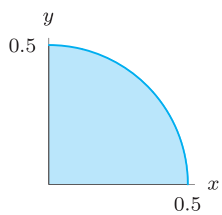
$$\int_{\pi/2}^{\pi}\int_{1}^{2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Exercises and Problems for Section 16.4

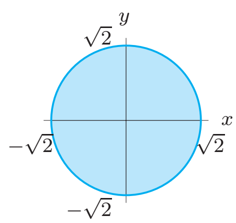
Exercises

For the regions R in Exercises 1–4, write $\int_R f\, dA$ as an iterated integral in polar coordinates.

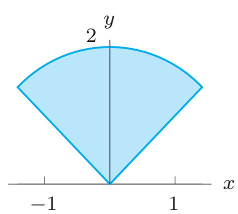
1.



2.



3.



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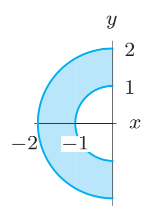
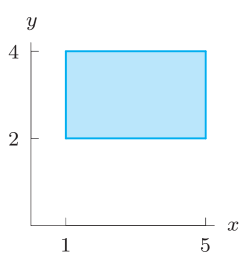


In Exercises 5–8, choose rectangular or polar coordinates to set up an iterated integral of an arbitrary function $f(x, y)$ over the region.

5.



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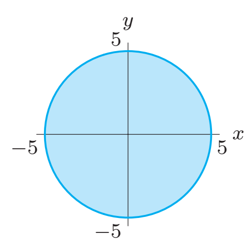

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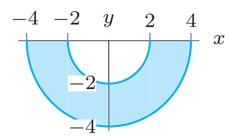
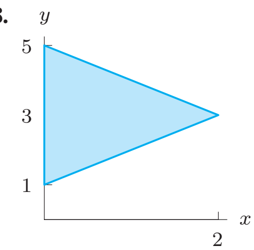


8.



Sketch the region of integration in Exercises 9–15.

9. $\displaystyle\int_0^4\int_{-\pi/2}^{\pi/2} f(r,\theta)\, r\, d\theta\, dr$

10. $\displaystyle\int_{\pi/2}^{\pi}\int_0^1 f(r,\theta)\, r\, dr\, d\theta$

11. $\displaystyle\int_0^{2\pi}\int_1^2 f(r,\theta)\, r\, dr\, d\theta$

12. $\displaystyle\int_{\pi/6}^{\pi/3}\int_0^1 f(r,\theta)\, r\, dr\, d\theta$

13. $\displaystyle\int_0^{\pi/4}\int_0^{1/\cos\theta} f(r,\theta)\, r\, dr\, d\theta$

14. $\displaystyle\int_3^4\int_{3\pi/4}^{3\pi/2} f(r,\theta)\, r\, d\theta\, dr$

15. $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^{2/\sin\theta} f(r,\theta)\, r\, dr\, d\theta$

Problems

In Exercises 16–18, evaluate the integral.

16. $\int_R \sqrt{x^2+y^2}\, dxdy$ where R is $4 \le x^2+y^2 \le 9$.

17. $\int_R \sin(x^2+y^2)\, dA$, where R is the disk of radius 2 centered at the origin.

18. $\int_R (x^2-y^2)\, dA$, where R is the first quadrant region between the circles of radius 1 and radius 2.

Convert the integrals in Problems 19–21 to polar coordinates and evaluate.

19. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\, dy\, dx$

20. $\displaystyle\int_0^{\sqrt{6}}\int_{-x}^{x} dy\, dx$

21. $\displaystyle\int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} xy\, dx\, dy$

22. Consider the integral $\int_0^3 \int_{x/3}^1 f(x,y)\,dy\,dx$.

(a) Sketch the region R over which the integration is being performed.
(b) Rewrite the integral with the order of integration reversed.
(c) Rewrite the integral in polar coordinates.

23. **(a)** Use integration in the following coordinates to find the volume of a solid orange wedge with $x \geq 0$ and cut out by the planes $y = 0$, $y = x/\sqrt{3}$, and a sphere of radius 5 centered at the origin. Which coordinates are the most efficient?

(i) Spherical coordinates
(ii) Cylindrical coordinates, in two different ways

(b) Calculate the volume without integration

24. Evaluate the integral by converting it into Cartesian coordinates:

$$\int_0^{\pi/6} \int_0^{2/\cos\theta} r\,dr\,d\theta.$$

25. **(a)** Sketch the region of integration of

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\,dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} x\,dy\,dx$$

(b) Evaluate the quantity in part (a).

26. Find the volume of the region between the graph of $f(x,y) = 25 - x^2 - y^2$ and the xy plane.

27. Find the volume of an ice cream cone bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

28. **(a)** For $a > 0$, find the volume under the graph of $z = e^{-(x^2+y^2)}$ above the disk $x^2 + y^2 \leq a^2$.
(b) What happens to the volume as $a \to \infty$?

29. A circular metal disk of radius 3 lies in the xy-plane with its center at the origin. At a distance r from the origin, the density of the metal per unit area is $\delta = \dfrac{1}{r^2+1}$.

(a) Write a double integral giving the total mass of the disk. Include limits of integration.
(b) Evaluate the integral.

30. A city surrounds a bay as shown in Figure 16.36. The population density of the city (in thousands of people per square km) is $\delta(r,\theta)$, where r and θ are polar coordinates and distances are in km.

(a) Set up an iterated integral in polar coordinates giving the total population of the city.
(b) The population density decreases the farther you live from the shoreline of the bay; it also decreases the farther you live from the ocean. Which of the following functions best describes this situation?

(i) $\delta(r,\theta) = (4-r)(2+\cos\theta)$
(ii) $\delta(r,\theta) = (4-r)(2+\sin\theta)$
(iii) $\delta(r,\theta) = (r+4)(2+\cos\theta)$

(c) Estimate the population using your answers to parts (a) and (b).

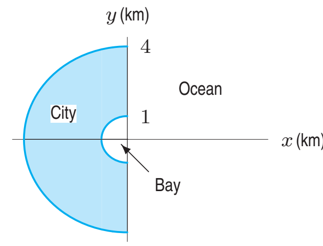


Figure 16.36

31. A disk of radius 5 cm has density 10 gm/cm^2 at its center and density 0 at its edge, and its density is a linear function of the distance from the center. Find the mass of the disk.

32. Electric charge is distributed over the xy-plane, with density inversely proportional to the distance from the origin. Show that the total charge inside a circle of radius R centered at the origin is proportional to R. What is the constant of proportionality?

33. **(a)** Graph $r = 1/(2\cos\theta)$ for $-\pi/2 \leq \theta \leq \pi/2$ and $r = 1$.
(b) Write an iterated integral representing the area inside the curve $r = 1$ and to the right of $r = 1/(2\cos\theta)$. Evaluate the integral.

34. **(a)** Sketch the circles $r = 2\cos\theta$ for $-\pi/2 \leq \theta \leq \pi/2$ and $r = 1$.
(b) Write an iterated integral representing the area inside the circle $r = 2\cos\theta$ and outside the circle $r = 1$. Evaluate the integral.

35. Two circular disks, each of radius 1, have centers which are 1 unit apart. Write, but do not evaluate, a double integral, including limits of integration, giving the area of overlap of the disks in

(a) Cartesian coordinates **(b)** Polar coordinates

36. Find the area inside the curve $r = 2 + 3\cos\theta$ and outside the circle $r = 2$.

Strengthen Your Understanding

In Problems 37–38, explain what is wrong with the statement.

37. If R is the region bounded by $x = 1, y = 0, y = x$, then in polar coordinates $\int_R x\,dA = \int_0^{\pi/4} \int_0^1 r^2 \cos\theta\,dr\,d\theta$.

38. If R is the region $x^2 + y^2 \leq 4$, then $\int_R (x^2 + y^2)\,dA = \int_0^{2\pi} \int_0^2 r^2\,dr\,d\theta$.