Solution We break the cone in Figure 16.25 into small cubes of volume $\Delta V = \Delta x\, \Delta y\, \Delta z$, on which the density is approximately constant, and approximate the mass of each cube by $\delta(x, y, z)\, \Delta x\, \Delta y\, \Delta z$. Stacking the cubes vertically above the point $(x, y, 0)$, starting on the cone at height $z = \sqrt{x^2 + y^2}$ and going up to $z = 3$, tells us that the inner integral is

$$\int_{\sqrt{x^2+y^2}}^{3} \delta(x, y, z)\, dz = \int_{\sqrt{x^2+y^2}}^{3} z\, dz.$$

There is a stack for every point in the xy-plane in the shadow of the cone. The cone $z = \sqrt{x^2 + y^2}$ intersects the horizontal plane $z = 3$ in the circle $x^2 + y^2 = 9$, so there is a stack for all (x, y) in the region $x^2 + y^2 \le 9$. Lining up the stacks parallel to the y-axis gives a slice from $y = -\sqrt{9 - x^2}$ to $y = \sqrt{9 - x^2}$, for each fixed value of x. Thus, the limits on the middle integral are

$$\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z\, dz\, dy.$$

Finally, there is a slice for each x between -3 and 3, so the integral we want is

$$\text{Mass} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{3} z\, dz\, dy\, dx.$$

Notice that setting up the limits on the two outer integrals was just like setting up the limits for a double integral over the region $x^2 + y^2 \le 9$.

As the previous example illustrates, for a region W contained between two surfaces, the innermost limits correspond to these surfaces. The middle and outer limits ensure that we integrate over the "shadow" of W in the xy-plane.

Limits on Triple Integrals

- The limits for the outer integral are constants.
- The limits for the middle integral can involve only one variable (that in the outer integral).
- The limits for the inner integral can involve two variables (those on the two outer integrals).

Exercises and Problems for Section 16.3

Exercises

In Exercises 1–4, find the triple integrals of the function over the region W.

1. $f(x, y, z) = x^2 + 5y^2 - z$, W is the rectangular box $0 \le x \le 2$, $-1 \le y \le 1$, $2 \le z \le 3$.
2. $h(x, y, z) = ax + by + cz$, W is the rectangular box $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2$.
3. $f(x, y, z) = \sin x \cos(y+z)$, W is the cube $0 \le x \le \pi$, $0 \le y \le \pi$, $0 \le z \le \pi$.
4. $f(x, y, z) = e^{-x-y-z}$, W is the rectangular box with corners at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$.

Sketch the region of integration in Exercises 5–13.

5. $\displaystyle\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x, y, z)\, dz\, dx\, dy$
6. $\displaystyle\int_0^1 \int_{-1}^1 \int_0^{\sqrt{1-z^2}} f(x, y, z)\, dy\, dz\, dx$
7. $\displaystyle\int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x, y, z)\, dz\, dx\, dy$

8. $\displaystyle\int_{-1}^{1}\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} f(x,y,z)\,dy\,dz\,dx$

9. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$

10. $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{0}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

11. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz\,dx\,dy$

12. $\displaystyle\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$

13. $\displaystyle\int_{0}^{1}\int_{0}^{\sqrt{1-z^2}}\int_{-\sqrt{1-x^2-z^2}}^{\sqrt{1-x^2-z^2}} f(x,y,z)\,dy\,dx\,dz$

Problems

In Problems 14–18, decide whether the integrals are positive, negative, or zero. Let S be the solid sphere $x^2+y^2+z^2 \le 1$, and T be the top half of this sphere (with $z \ge 0$), and B be the bottom half (with $z \le 0$), and R be the right half of the sphere (with $x \ge 0$), and L be the left half (with $x \le 0$).

14. $\displaystyle\int_T e^z\,dV$

15. $\displaystyle\int_B e^z\,dV$

16. $\displaystyle\int_S \sin z\,dV$

17. $\displaystyle\int_T \sin z\,dV$

18. $\displaystyle\int_R \sin z\,dV$

Let W be the solid cone bounded by $z = \sqrt{x^2+y^2}$ and $z = 2$. For Problems 19–27, decide (without calculating its value) whether the integral is positive, negative, or zero.

19. $\int_W y\,dV$

20. $\int_W x\,dV$

21. $\int_W z\,dV$

22. $\int_W xy\,dV$

23. $\int_W xyz\,dV$

24. $\int_W (z-2)\,dV$

25. $\int_W \sqrt{x^2+y^2}\,dV$

26. $\int_W e^{-xyz}\,dV$

27. $\int_W (z-\sqrt{x^2+y^2})\,dV$

28. Find the volume of the region bounded by the planes $z = 3y$, $z = y$, $y = 1$, $x = 1$, and $x = 2$.

29. Find the volume of the region bounded by $z = x^2$, $0 \le x \le 5$, and the planes $y = 0$, $y = 3$, and $z = 0$.

30. Find the volume of the region in the first octant bounded by the coordinate planes and the surface $x + y + z = 2$.

31. A trough with triangular cross-section lies along the x-axis for $0 \le x \le 10$. The slanted sides are given by $z = y$ and $z = -y$ for $0 \le z \le 1$ and the ends by $x = 0$ and $x = 10$, where x, y, z are in meters. The trough contains a sludge whose density at the point (x, y, z) is $\delta = e^{-3x}$ kg per m^3.

(a) Express the total mass of sludge in the trough in terms of triple integrals.

(b) Find the mass.

32. Find the volume of the region bounded by $z = x+y$, $z = 10$, and the planes $x = 0$, $y = 0$.

In Problems 33–38, write a triple integral, including limits of integration, that gives the specified volume.

33. Between $z = x + y$ and $z = 1 + 2x + 2y$ and above $0 \le x \le 1, 0 \le y \le 2$.

34. Between the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 4$ and above the disk $x^2 + y^2 \le 1$.

35. Between $2x + 2y + z = 6$ and $3x + 4y + z = 6$ and above $x + y \le 1, x \ge 0, y \ge 0$.

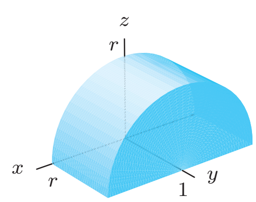
36. Under the sphere $x^2 + y^2 + z^2 = 9$ and above the region between $y = x$ and $y = 2x - 2$ in the xy-plane in the first quadrant.

37. Between the top portion of the sphere $x^2 + y^2 + z^2 = 9$ and the plane $z = 2$.

38. Under the sphere $x^2 + y^2 + z^2 = 4$ and above the region $x^2 + y^2 \le 4, 0 \le x \le 1, 0 \le y \le 2$ in the xy-plane.

In Problems 39–42, write limits of integration for the integral $\int_W f(x, y, z)\,dV$ where W is the quarter or half sphere or cylinder shown.

39.



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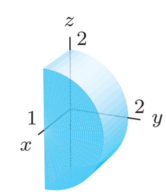

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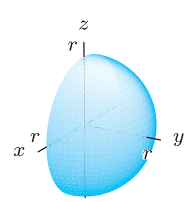

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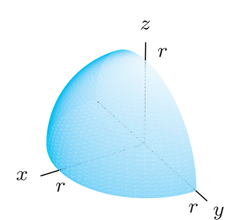

43. Find the volume of the region between the plane $z = x$ and the surface $z = x^2$, and the planes $y = 0$, and $y = 3$.

44. Find the volume of the region bounded by $z = x + y$, $0 \le x \le 5$, $0 \le y \le 5$, and the planes $x = 0$, $y = 0$, and $z = 0$.

45. Find the volume of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$ and $y - x = 4$ and $2x + y + z = 4$.

46. Find the volume between the planes $z = 1 + x + y$ and $x + y + z = 1$ and above the triangle $x + y \le 1, x \ge 0$, $y \ge 0$ in the xy-plane.

47. Find the volume between the plane $x + y + z = 1$ and the xy-plane, for $x + y \le 2$, $x \ge 0, y \ge 0$.

48. A solid shaped like a wedge of cheese has as its base the xy-plane, bounded by the x-axis, the line $y = x$ and the line $x + y = 1$. Its sides are vertical, and its top is the plane $x + y + z = 2$. At any point, the density of the solid is four times the distance from the xy-plane.

(a) Express the mass of the region in terms of triple integrals.
(b) Find the mass.

49. Find the mass of a triangular-shaped solid bounded by the planes $z = 1 + x$, $z = 1 - x$, $z = 0$, and with $0 \le y \le 3$. The density is $\delta = 10 - z$ gm/(cm)^3, and x, y, z are in cm.

50. Find the mass of the solid bounded by the xy-plane, yz-plane, xz-plane, and the plane $(x/3) + (y/2) + (z/6) = 1$, if the density of the solid is given by $\delta(x, y, z) = x + y$.

51. Find the mass of the pyramid with base in the plane $z = -6$ and sides formed by the three planes $y = 0$ and $y - x = 4$ and $2x + y + z = 4$, if the density of the solid is given by $\delta(x, y, z) = y$.

52. Let E be the solid pyramid bounded by the planes $x + z = 6$, $x - z = 0$, $y + z = 6$, $y - z = 0$, and above the plane $z = 0$ (see Figure 16.26). The density at any point in the pyramid is given by $\delta(x, y, z) = z$ grams per cm^3, where x, y, and z are measured in cm.

(a) Explain in practical terms what the triple integral $\int_E z\,dV$ represents.
(b) In evaluating the integral from part (a), how many separate triple integrals would be required if we chose to integrate in the z-direction first?
(c) Evaluate the triple integral from part (a) by integrating in a well-chosen order.

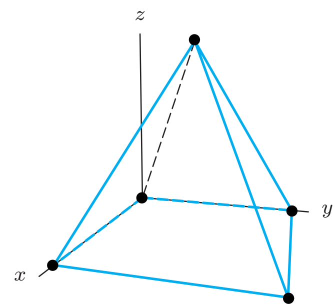

Figure 16.26

53. **(a)** What is the equation of the plane passing through the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$?
(b) Find the volume of the region bounded by this plane and the planes $x = 0$, $y = 0$, and $z = 0$.

Problems 54–56 refer to Figure 16.27, which shows triangular portions of the planes $2x + 4y + z = 4$, $3x - 2y = 0$, $z = 2$, and the three coordinate planes $x = 0, y = 0$, and $z = 0$. For each solid region E, write down an iterated integral for the triple integral $\int_E f(x, y, z)\,dV$.

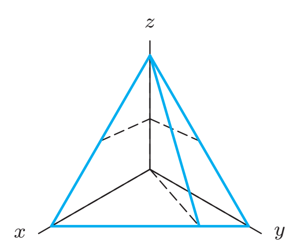

Figure 16.27

54. E is the region bounded by $y = 0$, $z = 0$, $3x - 2y = 0$, and $2x + 4y + z = 4$.

55. E is the region bounded by $x = 0$, $y = 0$, $z = 0$, $z = 2$, and $2x + 4y + z = 4$.

56. E is the region bounded by $x = 0$, $z = 0$, $3x - 2y = 0$, and $2x + 4y + z = 4$.

57. Figure 16.28 shows part of a spherical ball of radius 5 cm. Write an iterated triple integral which represents the volume of this region.

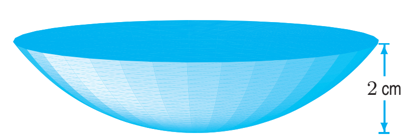

Figure 16.28