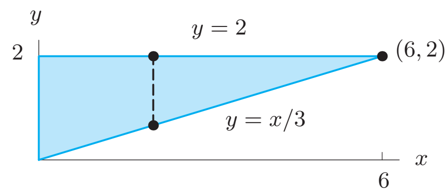


Figure 16.19: The region of integration for Example 6, showing the vertical strip

Reversing the Order of Integration

It is sometimes helpful to reverse the order of integration in an iterated integral. An integral which is difficult or impossible with the integration in one order can be quite straightforward in the other. The next example is such a case.

Example 7 Evaluate $\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\; dy\, dx$ using the region sketched in Figure 16.19.

Solution Since $\sqrt{y^3+1}$ has no elementary antiderivative, we cannot calculate the inner integral symbolically. We try reversing the order of integration. From Figure 16.19, we see that horizontal strips go from $x = 0$ to $x = 3y$ and that there is a strip for every y from 0 to 2. Thus, when we change the order of integration we get

$$\int_0^6 \int_{x/3}^2 x\sqrt{y^3+1}\, dy\, dx = \int_0^2 \int_0^{3y} x\sqrt{y^3+1}\, dx\, dy.$$

Now we can at least do the inner integral because we know the antiderivative of x. What about the outer integral?

$$\int_0^2 \int_0^{3y} x\sqrt{y^3+1}\, dx\, dy = \int_0^2 \left(\frac{x^2}{2}\sqrt{y^3+1} \right)\Bigg|_{x=0}^{x=3y} dy = \int_0^2 \frac{9y^2}{2}(y^3+1)^{1/2}\, dy$$
$$= (y^3+1)^{3/2}\Big|_0^2 = 27 - 1 = 26.$$

Thus, reversing the order of integration made the integral in the previous problem much easier. Notice that to reverse the order it is essential first to sketch the region over which the integration is being performed.

Exercises and Problems for Section 16.2

Exercises

In Exercises 1–4, sketch the region of integration.

1. $\int_0^\pi \int_0^x y \sin x\, dy\, dx$

2. $\int_0^1 \int_{y^2}^y xy\, dx\, dy$

3. $\int_0^2 \int_0^{y^2} y^2 x\, dx\, dy$

4. $\int_0^1 \int_{x-2}^{\cos \pi x} y\, dy\, dx$

For Exercises 5–12, evaluate the integral.

5. $\int_0^3 \int_0^4 (4x+3y)\, dx\, dy$

6. $\int_0^2 \int_0^3 (x^2+y^2)\, dy\, dx$

7. $\int_0^3 \int_0^2 6xy\, dy\, dx$

8. $\int_0^1 \int_0^2 x^2 y\, dy\, dx$

9. $\int_0^1 \int_0^1 ye^{xy}\, dx\, dy$

10. $\int_0^2 \int_0^y y\, dx\, dy$

11. $\int_0^3 \int_0^y \sin x\, dx\, dy$

12. $\int_0^{\pi/2} \int_0^{\sin x} x\, dy\, dx$

For Exercises 13–16, sketch the region of integration and evaluate the integral.

13. $\int_1^3 \int_0^4 e^{x+y}\, dy\, dx$

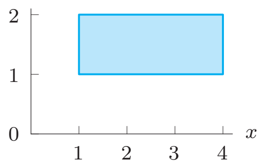
14. $\int_0^2 \int_0^x e^{x^2}\, dy\, dx$

15. $\int_1^5 \int_x^{2x} \sin x\, dy\, dx$

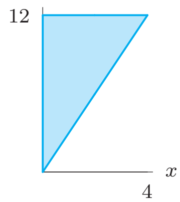
16. $\int_1^4 \int_{\sqrt{y}}^y x^2 y^3\, dx\, dy$

In Exercises 17–22, write $\int_R f\,dA$ as an iterated integral for the shaded region R.

17.



18.



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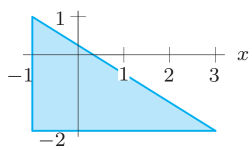

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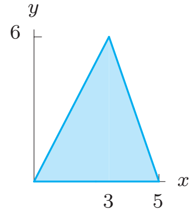

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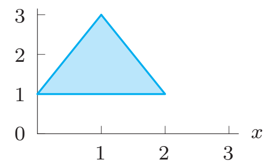

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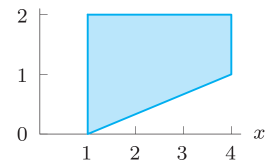


For Exercises 23–27, evaluate the integral.

23. $\int_R \sqrt{x+y}\,dA$, where R is the rectangle $0 \le x \le 1$, $0 \le y \le 2$.

24. Calculate the integral in Exercise 23 using the other order of integration.

25. $\int_R (5x^2+1)\sin 3y\,dA$, where R is the rectangle $-1 \le x \le 1$, $0 \le y \le \pi/3$.

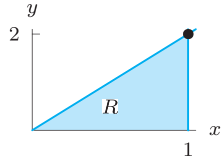
26. $\int_R xy\,dA$, where R is the triangle $x+y \le 1$, $x \ge 0$, $y \ge 0$.

27. $\int_R (2x+3y)^2\,dA$, where R is the triangle with vertices at $(-1,0)$, $(0,1)$, and $(1,0)$.

Problems

In Problems 28–31, integrate $f(x,y) = xy$ over the region R.

28.



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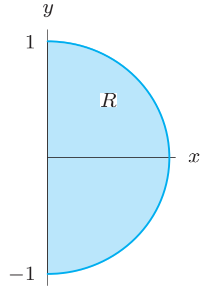

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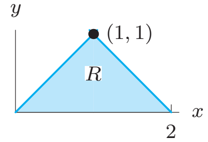

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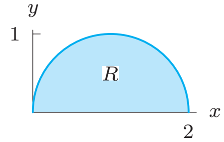

32. **(a)** Use four subrectangles to approximate the volume of the object whose base is the region $0 \le x \le 4$ and $0 \le y \le 6$, and whose height is given by $f(x,y) = xy$. Find an overestimate and an underestimate and average the two.

(b) Integrate to find the exact volume of the three-dimensional object described in part (a).

In Problems 33–37, evaluate the integral by reversing the order of integration.

33. $\displaystyle\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$

34. $\displaystyle\int_0^1 \int_y^1 \sin\left(x^2\right)\,dx\,dy$

35. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{2+x^3}\,dx\,dy$

36. $\displaystyle\int_0^3 \int_{y^2}^9 y\sin(x^2)\,dx\,dy$

37. $\displaystyle\int_0^1 \int_{e^y}^e \frac{x}{\ln x}\,dx\,dy$

38. Find the volume under the graph of the function $f(x,y) = 6x^2y$ over the region shown in Figure 16.20.

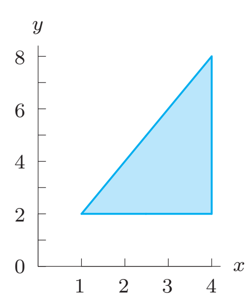

Figure 16.20

39. **(a)** Find the volume below the surface $z = x^2+y^2$ and above the xy-plane for $-1 \le x \le 1, -1 \le y \le 1$.

(b) Find the volume above the surface $z = x^2+y^2$ and below the plane $z = 2$ for $-1 \le x \le 1$, $-1 \le y \le 1$.

40. Compute the integral

$$\int\int_R (2x^2 + y)\, dA,$$

where R is the triangular region with vertices at $(0, 1)$, $(-2, 3)$ and $(2, 3)$.

41. **(a)** Sketch the region in the xy-plane bounded by the x-axis, $y = x$, and $x + y = 1$.
(b) Express the integral of $f(x, y)$ over this region in terms of iterated integrals in two ways. (In one, use $dx\, dy$; in the other, use $dy\, dx$.)
(c) Using one of your answers to part (b), evaluate the integral exactly with $f(x, y) = x$.

42. Let $f(x, y) = x^2 e^{x^2}$ and let R be the triangle bounded by the lines $x = 3$, $x = y/2$, and $y = x$ in the xy-plane.

(a) Express $\int_R f\, dA$ as a double integral in two different ways.
(b) Evaluate one of them.

43. Find the average value of $f(x, y) = x^2 + 4y$ on the rectangle $0 \le x \le 3$ and $0 \le y \le 6$.

44. Find the average value of $f(x, y) = xy^2$ on the rectangle $0 \le x \le 4$, $0 \le y \le 3$.

In Problems 45–47 set up, but do not evaluate, an iterated integral for the volume of the solid.

45. Under the graph of $f(x, y) = 25 - x^2 - y^2$ and above the xy-plane.

46. Below the graph of $f(x, y) = 25 - x^2 - y^2$ and above the plane $z = 16$.

47. The three-sided pyramid whose base is on the xy-plane and whose three sides are the vertical planes $y = 0$ and $y - x = 4$, and the slanted plane $2x + y + z = 4$.

In Problems 48–53, find the volume of the solid region.

48. Under the graph of $f(x, y) = xy$ and above the square $0 \le x \le 2$, $0 \le y \le 2$ in the xy-plane.

49. Under the graph of $f(x, y) = x^2 + y^2$ and above the triangle $0 \le y \le x, 0 \le x \le 1$.

50. Under the graph of $f(x, y) = x + y$ and above the region $y^2 \le x, 0 \le x \le 9, y \ge 0$.

51. Under the graph of $2x + y + z = 4$ in the first octant.

52. The solid between the planes $z = 3x + 2y + 1$ and $z = x + y$, and above the triangle with vertices $(1, 0, 0)$, $(2, 2, 0)$, and $(0, 1, 0)$ in the xy-plane. See Figure 16.21.

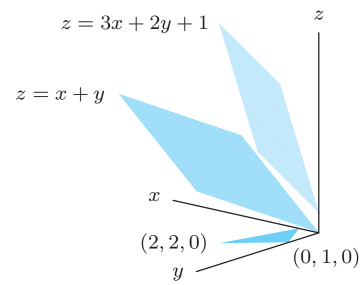

Figure 16.21

53. The solid region R bounded by the coordinate planes and the graph of $ax + by + cz = 1$. Assume a, b, and $c > 0$.

54. If R is the region $x + y \ge a, x^2 + y^2 \le a^2$, with $a > 0$, evaluate the integral

$$\int_R xy\, dA.$$

55. The region W lies below the surface $f(x, y) = 2e^{-(x-1)^2-y^2}$ and above the disk $x^2 + y^2 \le 4$ in the xy-plane.

(a) Describe in words the contours of f, using $f(x, y) = 1$ as an example.
(b) Write an integral giving the area of the cross-section of W in the plane $x = 1$.
(c) Write an iterated double integral giving the volume of W.

56. Find the average distance to the x-axis for points in the region bounded by the x-axis and the graph of $y = x - x^2$.

57. Give the contour diagram of a function f whose average value on the square $0 \le x \le 1, 0 \le y \le 1$ is

(a) Greater than the average of the values of f at the four corners of the square.
(b) Less than the average of the values of f at the four corners of the square.

58. The function $f(x, y) = ax + by$ has an average value of 20 on the rectangle $0 \le x \le 2, 0 \le y \le 3$.

(a) What can you say about the constants a and b?
(b) Find two different choices for f that have average value 20 on the rectangle, and give their contour diagrams on the rectangle.

59. The function $f(x, y) = ax^2 + bxy + cy^2$ has an average value of 20 on the square $0 \le x \le 2, 0 \le y \le 2$.

(a) What can you say about the constants a, b, and c?
(b) Find two different choices for f that have average value 20 on the square, and give their contour diagrams on the square.