we are likely to meet, the area of the subrectangles covering the edge tends to 0 as the grid becomes finer. Therefore, omitting these rectangles does not affect the limit.

Convergence of Upper and Lower Sums to Same Limit

We have said that if f is continuous on the rectangle R, then the difference between upper and lower sums for f converges to 0 as Δx and Δy approach 0. In the following example, we show this in a particular case. The ideas in this example can be used in a general proof.

Example 3 Let $f(x, y) = x^2y$ and let R be the rectangle $0 \le x \le 1$, $0 \le y \le 1$. Show that the difference between upper and lower Riemann sums for f on R converges to 0, as Δx and Δy approach 0.

Solution The difference between the sums is

$$\sum M_{ij}\Delta x\Delta y - \sum L_{ij}\,\Delta x\Delta y = \sum(M_{ij} - L_{ij})\,\Delta x\Delta y,$$

where M_{ij} and L_{ij} are the maximum and minimum of f on the ij-th subrectangle. Since f increases in both the x and y directions, M_{ij} occurs at the corner of the subrectangle farthest from the origin and L_{ij} at the closest. Moreover, since the slopes in the x and y directions don't decrease as x and y increase, the difference $M_{ij} - L_{ij}$ is largest in the subrectangle R_{nm} which is farthest from the origin. Thus,

$$\sum(M_{ij} - L_{ij})\,\Delta x\Delta y \le (M_{nm} - L_{nm})\sum\Delta x\Delta y = (M_{nm} - L_{nm})\text{Area}(R).$$

Thus, the difference converges to 0 as long as $(M_{nm} - L_{nm})$ does. The maximum M_{nm} of f on the nm-th subrectangle occurs at $(1, 1)$, the subrectangle's top right corner, and the minimum L_{nm} occurs at the opposite corner, $(1 - 1/n, 1 - 1/m)$. Substituting into $f(x, y) = x^2y$ gives

$$M_{nm} - L_{nm} = (1)^2(1) - \left(1 - \frac{1}{n}\right)^2\left(1 - \frac{1}{m}\right) = \frac{2}{n} - \frac{1}{n^2} + \frac{1}{m} - \frac{2}{nm} + \frac{1}{n^2m}.$$

The right-hand side converges to 0 as $n, m \to \infty$, that is, as $\Delta x, \Delta y \to 0$.

Exercises and Problems for Section 16.1

Exercises

1. Table 16.4 gives values of the function $f(x, y)$, which is increasing in x and decreasing in y on the region $R: 0 \le x \le 6$, $0 \le y \le 1$. Make the best possible upper and lower estimates of $\int_R f(x, y)\,dA$.

Table 16.4

y \ x	0	3	6
0	5	7	10
0.5	4	5	7
1	3	4	6

2. Values of $f(x, y)$ are in Table 16.5. Let R be the rectangle $1 \le x \le 1.2$, $2 \le y \le 2.4$. Find Riemann sums which are reasonable over and underestimates for $\int_R f(x, y)\,dA$ with $\Delta x = 0.1$ and $\Delta y = 0.2$.

Table 16.5

y \ x	1.0	1.1	1.2
2.0	5	7	10
2.2	4	6	8
2.4	3	5	4

3. Figure 16.6 shows contours of $g(x, y)$ on the region R, with $5 \le x \le 11$ and $4 \le y \le 10$. Using $\Delta x = \Delta y = 2$, find an overestimate and an underestimate for $\int_R g(x, y)dA$.

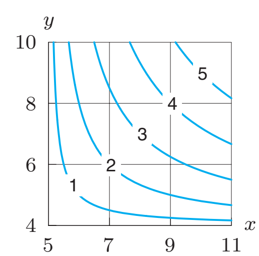

Figure 16.6

4. Figure 16.7 shows contours of $f(x, y)$ on the rectangle R with $0 \le x \le 30$ and $0 \le y \le 15$. Using $\Delta x = 10$ and $\Delta y = 5$, find an overestimate and an underestimate for $\int_R f(x, y)dA$.

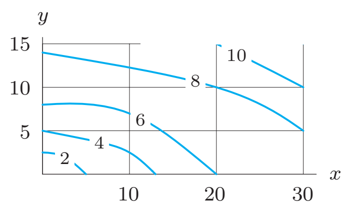

Figure 16.7

5. Figure 16.8 shows a contour plot of population density, people per square kilometer, in a rectangle of land 3 km by 2 km. Estimate the population in the region represented by Figure 16.8.

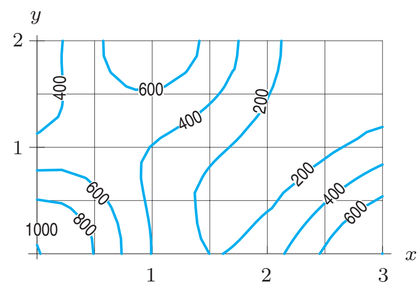

Figure 16.8

Problems

In Problems 6–12, decide (without calculation) whether the integrals are positive, negative, or zero. Let D be the region inside the unit circle centered at the origin, let R be the right half of D and let B be the bottom half of D.

6. $\int_D dA$

7. $\int_R 5x\, dA$

8. $\int_B 5x\, dA$

9. $\int_D (y^3 + y^5)\, dA$

10. $\int_B (y^3 + y^5)\, dA$

11. $\int_D (y - y^3)\, dA$

12. $\int_B (y - y^3)\, dA$

13. Figure 16.9 shows contours of $f(x, y)$. Let R be the square $-0.5 \le x \le 1$, $-0.5 \le y \le 1$. Is the integral $\int_R f\, dA$ positive or negative? Explain your reasoning.

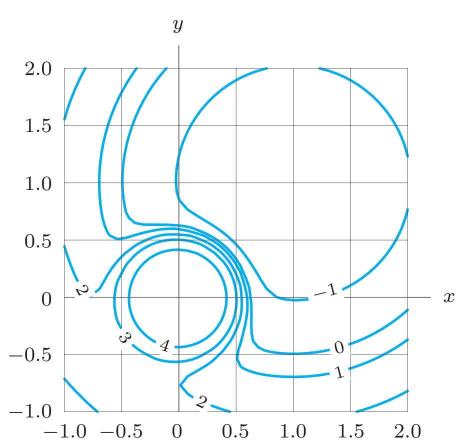

Figure 16.9

14. Table 16.6 gives values of $f(x, y)$, the number of milligrams of mosquito larvae per square meter in a swamp. If x and y are in meters and R is the rectangle $0 \le x \le 8$, $0 \le y \le 6$, estimate $\int_R f(x, y)dA$. Give units and interpret your answer.

Table 16.6

		x		
		0	4	8
y	0	1	3	6
	3	2	5	9
	6	4	9	15

15. Figure 16.10 shows the temperature, in °C, in a 5 meter by 5 meter heated room. Using Riemann sums, estimate the average temperature in the room.

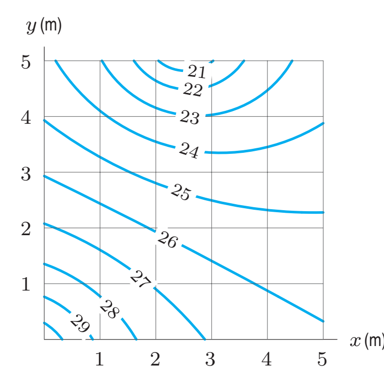

Figure 16.10

16. Use four subrectangles to approximate the volume of the object whose base is the region $0 \le x \le 4$ and $0 \le y \le 6$, and whose height is given by $f(x, y) = x + y$. Find an overestimate and an underestimate and average the two.