and solve the system of equations we get from $\operatorname{grad}\mathcal{L} = \vec{0}$:

$$\frac{\partial \mathcal{L}}{\partial x} = 20x^{-3/5}y^{1/5}z^{1/5} - 80\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial y} = 10x^{2/5}y^{-4/5}z^{1/5} - 12\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial z} = 10x^{2/5}y^{1/5}z^{-4/5} - 10\lambda = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -(80x + 12y + 10z - 24{,}000) = 0.$$

Simplifying this system gives

$$\lambda = \frac{1}{4}x^{-3/5}y^{1/5}z^{1/5},$$
$$\lambda = \frac{5}{6}x^{2/5}y^{-4/5}z^{1/5},$$
$$\lambda = x^{2/5}y^{1/5}z^{-4/5},$$
$$80x + 12y + 10z = 24{,}000.$$

Eliminating z from the first two equations gives $x = 0.3y$. Eliminating x from the second and third equations gives $z = 1.2y$. Substituting for x and z into $80x + 12y + 10z = 24{,}000$ gives

$$80(0.3y) + 12y + 10(1.2y) = 24{,}000,$$

so $y = 500$. Then $x = 150$ and $z = 600$, and $f(150, 500, 600) = 4{,}622$ units.

The graph of the constraint, $80x + 12y + 10z = 24{,}000$, is a plane. Since the inputs x, y, z must be nonnegative, the graph is a triangle in the first octant, with edges on the coordinate planes. On the boundary of the triangle, one (or more) of the variables x, y, z is zero, so the function f is zero. Thus production is maximized within the budget using $x = 150$, $y = 500$, and $z = 600$.

Exercises and Problems for Section 15.3

Exercises

In Exercises 1–17, use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint, if such values exist.

1. $f(x, y) = x + y, \quad x^2 + y^2 = 1$

2. $f(x, y) = x + 3y + 2, \quad x^2 + y^2 = 10$

3. $f(x, y) = (x - 1)^2 + (y + 2)^2, \quad x^2 + y^2 = 5$

4. $f(x, y) = x^3 + y, \quad 3x^2 + y^2 = 4$

5. $f(x, y) = 3x - 2y, \quad x^2 + 2y^2 = 44$

6. $f(x, y) = 2xy, \quad 5x + 4y = 100$

7. $f(x_1, x_2) = {x_1}^2 + {x_2}^2, \quad x_1 + x_2 = 1$

8. $f(x, y) = x^2 + y, \quad x^2 - y^2 = 1$

9. $f(x, y, z) = x + 3y + 5z, \quad x^2 + y^2 + z^2 = 1$

10. $f(x, y, z) = x^2 - y^2 - 2z, \quad x^2 + y^2 = z$

11. $f(x, y, z) = xyz, \quad x^2 + y^2 + 4z^2 = 12$

12. $f(x, y) = x^2 + 2y^2, \quad x^2 + y^2 \le 4$

13. $f(x, y) = x + 3y, \quad x^2 + y^2 \le 2$

14. $f(x, y) = xy, \quad x^2 + 2y^2 \le 1$

15. $f(x, y) = x^3 + y, \quad x + y \ge 1$

16. $f(x, y) = (x + 3)^2 + (y - 3)^2, \quad x^2 + y^2 \le 2$

17. $f(x, y) = x^2y + 3y^2 - y, \quad x^2 + y^2 \le 10$

18. Decide whether each point appears to be a maximum, minimum, or neither for the function f constrained by the loop in Figure 15.30.

(a) P **(b)** Q **(c)** R **(d)** S

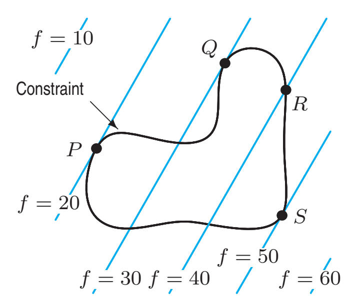

Figure 15.30