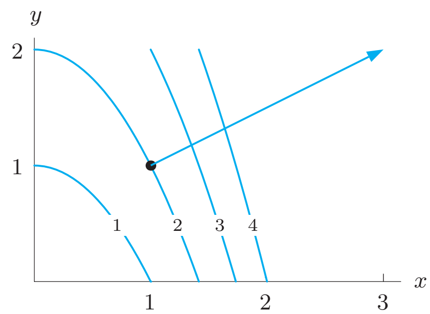

Figure 14.46: The gradient vector with x and y scales equal

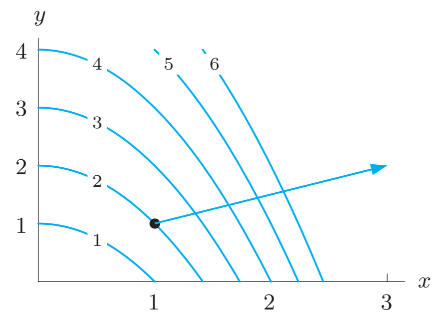

Figure 14.47: The gradient vector with x and y scales unequal

Exercises and Problems for Section 14.5

Exercises

In Exercises 1–12, find the gradient of the function.

1. $f(x, y, z) = x^2$

2. $f(x, y, z) = x^2 + y^3 - z^4$

3. $f(x, y, z) = e^{x+y+z}$

4. $f(x, y, z) = \cos(x + y) + \sin(y + z)$

5. $f(x, y, z) = yz^2/(1 + x^2)$

6. $f(x, y, z) = 1/(x^2 + y^2 + z^2)$

7. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

8. $f(x, y, z) = xe^y \sin z$

9. $f(x, y, z) = xy + \sin(e^z)$

10. $f(x_1, x_2, x_3) = x_1^2 x_2^3 x_3^4$

11. $f(p, q, r) = e^p + \ln q + e^{r^2}$

12. $f(x, y, z) = e^{z^2} + y \ln(x^2 + 5)$

In Exercises 13–18, find the gradient at the point.

13. $f(x, y, z) = zy^2$, at $(1, 0, 1)$

14. $f(x, y, z) = 2x + 3y + 4z$, at $(1, 1, 1)$

15. $f(x, y, z) = x^2 + y^2 - z^4$, at $(3, 2, 1)$

16. $f(x, y, z) = xyz$, at $(1, 2, 3)$

17. $f(x, y, z) = \sin(xy) + \sin(yz)$, at $(1, \pi, -1)$

18. $f(x, y, z) = x \ln(yz)$, at $(2, 1, e)$

In Exercises 19–24, find the directional derivative using $f(x, y, z) = xy + z^2$.

19. At $(1, 2, 3)$ in the direction of $\vec{i} + \vec{j} + \vec{k}$.

20. At $(1, 1, 1)$ in the direction of $\vec{i} + 2\vec{j} + 3\vec{k}$.

21. As you leave the point $(1, 1, 0)$ heading in the direction of the point $(0, 1, 1)$.

22. As you arrive at $(0, 1, 1)$ from the direction of $(1, 1, 0)$.

23. At the point $(2, 3, 4)$ in the direction of a vector making an angle of $3\pi/4$ with grad $f(2, 3, 4)$.

24. At the point $(2, 3, 4)$ in the direction of the maximum rate of change of f.

In Exercises 25–30, check that the point $(-1, 1, 2)$ lies on the given surface. Then, viewing the surface as a level surface for a function $f(x, y, z)$, find a vector normal to the surface and an equation for the tangent plane to the surface at $(-1, 1, 2)$.

25. $x^2 - y^2 + z^2 = 4$

26. $z = x^2 + y^2$

27. $y^2 = z^2 - 3$

28. $x^2 - xyz = 3$

29. $\cos(x + y) = e^{xz+2}$

30. $y = 4/(2x + 3z)$

31. For $f(x, y, z) = 3x^2y^2 + 2yz$, find the directional derivative at the point $(-1, 0, 4)$ in the direction of
(a) $\vec{i} - \vec{k}$ **(b)** $-\vec{i} + 3\vec{j} + 3\vec{k}$

32. If $f(x, y, z) = x^2 + 3xy + 2z$, find the directional derivative at the point $(2, 0, -1)$ in the direction of $2\vec{i} + \vec{j} - 2\vec{k}$.

33. **(a)** Let $f(x, y, z) = x^2 + y^2 - xyz$. Find grad f.
(b) Find the equation for the tangent plane to the surface $f(x, y, z) = 7$ at the point $(2, 3, 1)$.

34. Find the equation of the tangent plane at the point $(3, 2, 2)$ to $z = \sqrt{17 - x^2 - y^2}$.

35. Find the equation of the tangent plane to $z = 8/(xy)$ at the point $(1, 2, 4)$.

36. Find an equation of the tangent plane and of a normal vector to the surface $x = y^3z^7$ at the point $(1, -1, -1)$.

In Exercises 37–38, the gradient of f and a point P on the level surface $f(x, y, z) = 0$ are given. Find an equation for the tangent plane to the surface at the point P.

37. grad $f = yz\vec{i} + xz\vec{j} + xy\vec{k}$, $P = (1, 2, 3)$

38. grad $f = 2x\vec{i} + z^2\vec{j} + 2yz\vec{k}$, $P = (10, -10, 30)$

In Exercises 39–43, find an equation of the tangent plane to the surface at the given point.

39. $x^2 + y^2 + z^2 = 17$ at the point $(2, 3, 2)$

40. $x^2 + y^2 = 1$ at the point $(1, 0, 0)$

41. $z = 2x + y + 3$ at the point $(0, 0, 3)$

42. $3x^2 - 4xy + z^2 = 0$ at the point (a, a, a), where $a \neq 0$

43. $z = 9/(x + 4y)$ at the point where $x = 1$ and $y = 2$

Problems

44. Consider the surface $g(x, y) = 4 - x^2$. What is the relation between $\operatorname{grad} g(-1, -1)$ and a vector tangent to the path of steepest ascent at $(-1, -1, 3)$? Illustrate your answer with a sketch.

45. Match the functions $f(x, y, z)$ in (a)–(d) with the descriptions of their gradients in (I)–(IV). No reasons needed.

(a) $x^2 + y^2 + z^2$ **(b)** $x^2 + y^2$

(c) $\dfrac{1}{x^2 + y^2 + z^2}$ **(d)** $\dfrac{1}{x^2 + y^2}$

I Points radially outward from the z-axis.
II Points radially inward toward the z-axis.
III Points radially outward from the origin.
IV Points radially inward toward the origin.

46. Find the equation of the tangent plane at $(2, 3, 1)$ to the surface $x^2 + y^2 - xyz = 7$. Do this in two ways:

(a) Viewing the surface as the level set of a function of three variables, $F(x, y, z)$.
(b) Viewing the surface as the graph of a function of two variables $z = f(x, y)$.

47. Let $f(x, y, z) = x^2 + y^2 + z^2$. At the point $(1, 2, 1)$, find the rate of change of f in the direction perpendicular to the plane $x + 2y + 3z = 8$ and moving away from the origin.

48. Let $f(x, y) = \cos x \sin y$ and let S be the surface $z = f(x, y)$.

(a) Find a normal vector to the surface S at the point $(0, \pi/2, 1)$.
(b) What is the equation of the tangent plane to the surface S at the point $(0, \pi/2, 1)$?

49. Let $f(x, y, z) = \sin(x^2 + y^2 + z^2)$.

(a) Describe in words the shape of the level surfaces of f.
(b) Find $\operatorname{grad} f$.
(c) Consider the two vectors $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\operatorname{grad} f$ at a point (x, y, z) where $\sin(x^2+y^2+z^2) \neq 0$. What is (are) the possible values(s) of the angle between these vectors?

50. Each diagram (I) – (IV) in Figure 14.48 represents the level curves of a function $f(x, y)$. For each function f, consider the point above P on the surface $z = f(x, y)$ and choose from the lists which follow:

(a) A vector which could be the normal to the surface at that point;
(b) An equation which could be the equation of the tangent plane to the surface at that point.

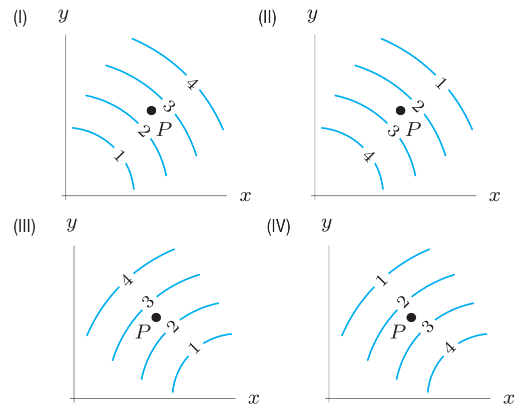

Figure 14.48

Vectors	Equations
(E) $2\vec{i} + 2\vec{j} - 2\vec{k}$	(J) $x + y + z = 4$
(F) $2\vec{i} + 2\vec{j} + 2\vec{k}$	(K) $2x - 2y - 2z = 2$
(G) $2\vec{i} - 2\vec{j} + 2\vec{k}$	(L) $-3x - 3y + 3z = 6$
(H) $-2\vec{i} + 2\vec{j} + 2\vec{k}$	(M) $-\dfrac{x}{2} + \dfrac{y}{2} - \dfrac{z}{2} = -7$

51. The surface S is represented by the equation $F = 0$ where $F(x, y, z) = x^2 - (y/z^2)$.

(a) Find the unit vectors $\vec{u}_1$ and $\vec{u}_2$ pointing in the direction of maximum increase of F at the points $(0, 0, 1)$ and $(1, 1, 1)$ respectively.
(b) Find the tangent plane to S at the points $(0, 0, 1)$ and $(1, 1, 1)$.
(c) Find all points on S where a normal vector is parallel to the xy-plane.

52. Consider the function $f(x, y) = (e^x - x)\cos y$. Suppose S is the surface $z = f(x, y)$.

(a) Find a vector which is perpendicular to the level curve of f through the point $(2, 3)$ in the direction in which f decreases most rapidly.
(b) Suppose $\vec{v} = 5\vec{i} + 4\vec{j} + a\vec{k}$ is a vector in 3-space which is tangent to the surface S at the point P lying on the surface above $(2, 3)$. What is a?

53. (a) Find the tangent plane to the surface $x^2+y^2+3z^2 = 4$ at the point $(0.6, 0.8, 1)$.
(b) Is there a point on the surface $x^2 + y^2 + 3z^2 = 4$ at which the tangent plane is parallel to the plane $8x + 6y + 30z = 1$? If so, find it. If not, explain why not.

54. Your house lies on the surface $z = f(x, y) = 2x^2 - y^2$ directly above the point $(4, 3)$ in the xy-plane.

(a) How high above the xy-plane do you live?
(b) What is the slope of your lawn as you look from your house directly toward the z-axis (that is, along the vector $-4\vec{i} - 3\vec{j}$)?

(c) When you wash your car in the driveway, on this surface above the point $(4, 3)$, which way does the water run off? (Give your answer as a two-dimensional vector.)

(d) What is the equation of the tangent plane to this surface at your house?

55. **(a)** Sketch the contours of $z = y - \sin x$ for $z = -1, 0, 1, 2$.

(b) A bug starts on the surface at the point $(\pi/2, 1, 0)$ and walks on the surface $z = y - \sin x$ in the direction parallel to the y-axis, in the direction of increasing y. Is the bug walking in a valley or on top of a ridge? Explain.

(c) On the contour $z = 0$ in your sketch for part (a), draw the gradients of z at $x = 0$, $x = \pi/2$, and $x = \pi$.

56. At what point on the surface $z = 1+x^2+y^2$ is its tangent plane parallel to the following planes?

(a) $z = 5$ **(b)** $z = 5 + 6x - 10y$.

57. The concentration of salt in a fluid at (x, y, z) is given by $F(x, y, z) = x^2 + y^4 + x^2z^2$ mg/cm^3. You are at the point $(-1, 1, 1)$.

(a) In which direction should you move if you want the concentration to increase the fastest?

(b) You start to move in the direction you found in part (a) at a speed of 4 cm/sec. How fast is the concentration changing?

58. Let $g_x(2, 1, 7) = 3$, $g_y(2, 1, 7) = 10$, $g_z(2, 1, 7) = -5$. Find the equation of the tangent plane to $g(x, y, z) = 0$ at the point $(2, 1, 7)$.

59. The vector ∇f at point P and four unit vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$ are shown in Figure 14.49. Arrange the following quantities in ascending order

$$f_{\vec{u}_1}, \quad f_{\vec{u}_2}, \quad f_{\vec{u}_3}, \quad f_{\vec{u}_4}, \quad \text{the number } 0.$$

The directional derivatives are all evaluated at the point P and the function $f(x, y)$ is differentiable at P.

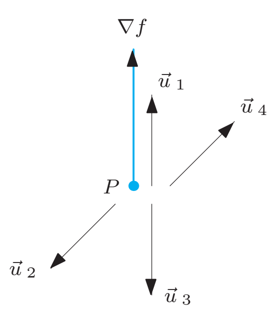

Figure 14.49

60. The temperature of a gas at the point (x, y, z) is given by $G(x, y, z) = x^2 - 5xy + y^2z$.

(a) What is the rate of change in the temperature at the point $(1, 2, 3)$ in the direction $\vec{v} = 2\vec{i} + \vec{j} - 4\vec{k}$?

(b) What is the direction of maximum rate of change of temperature at the point $(1, 2, 3)$?

(c) What is the maximum rate of change at the point $(1, 2, 3)$?

61. The temperature at the point (x, y, z) in 3-space is given, in degrees Celsius, by $T(x, y, z) = e^{-(x^2+y^2+z^2)}$.

(a) Describe in words the shape of surfaces on which the temperature is constant.

(b) Find grad T.

(c) You travel from the point $(1, 0, 0)$ to the point $(2, 1, 0)$ at a speed of 3 units per second. Find the instantaneous rate of change of the temperature as you leave the point $(1, 0, 0)$. Give units.

62. A spaceship is plunging into the atmosphere of a planet. With coordinates in miles and the origin at the center of the planet, the pressure of the atmosphere at (x, y, z) is

$$P = 5e^{-0.1\sqrt{x^2+y^2+z^2}} \text{ atmospheres.}$$

The velocity, in miles/sec, of the spaceship at $(0, 0, 1)$ is $\vec{v} = \vec{i} - 2.5\vec{k}$. At $(0, 0, 1)$, what is the rate of change with respect to time of the pressure on the spaceship?

63. Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $\vec{a}$ be a constant vector. For each of the quantities in (a)–(c), choose the statement in (I)–(V) that describes it. No reasons needed.

(a) $\text{grad}(\vec{r} + \vec{a})$ **(b)** $\text{grad}(\vec{r} \cdot \vec{a})$ **(c)** $\text{grad}(\vec{r} \times \vec{a})$

I Scalar, independent of $\vec{a}$.
II Scalar, depends on $\vec{a}$.
III Vector, independent of $\vec{a}$.
IV Vector, depends on $\vec{a}$.
V Not defined.

64. The earth has mass M and is located at the origin in 3-space, while the moon has mass m. Newton's Law of Gravitation states that if the moon is located at the point (x, y, z) then the attractive force exerted by the earth on the moon is given by the vector

$$\vec{F} = -GMm\frac{\vec{r}}{\|\vec{r}\|^3},$$

where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$. Show that $\vec{F} = \text{grad}\,\varphi$, where φ is the function given by

$$\varphi(x, y, z) = \frac{GMm}{\|\vec{r}\|}.$$

65. Two surfaces are said to be *tangential* at a point P if they have the same tangent plane at P. Show that the surfaces $z = \sqrt{2x^2 + 2y^2 - 25}$ and $z = \frac{1}{5}(x^2 + y^2)$ are tangential at the point $(4, 3, 5)$.

66. Two surfaces are said to be *orthogonal* to each other at a point P if the normals to their tangent planes are perpendicular at P. Show that the surfaces $z = \frac{1}{2}(x^2 + y^2 - 1)$ and $z = \frac{1}{2}(1 - x^2 - y^2)$ are orthogonal at all points of intersection.

67. Let $\vec{r}$ be the position vector of the point (x, y, z). If $\vec{\mu} = \mu_1\vec{i} + \mu_2\vec{j} + \mu_3\vec{k}$ is a constant vector, show that

$$\text{grad}(\vec{\mu} \cdot \vec{r}) = \vec{\mu}.$$

68. Let $\vec{r}$ be the position vector of the point (x, y, z). Show that, if a is a constant,

$$\text{grad}(\|\vec{r}\|^a) = a\|\vec{r}\|^{a-2}\vec{r}, \qquad \vec{r} \neq \vec{0}.$$

69. Let f and g be functions on 3-space. Suppose f is differentiable and that

$$\text{grad}\, f(x, y, z) = (x\vec{i} + y\vec{j} + z\vec{k})g(x, y, z).$$

Explain why f must be constant on any sphere centered at the origin.

Strengthen Your Understanding

In Problems 70–71, explain what is wrong with the statement.

70. The gradient vector grad $f(x, y)$ points in the direction perpendicular to the surface $z = f(x, y)$.

71. The tangent plane at the origin to a surface $f(x, y, z) = 1$ that contains the point $(0, 0, 0)$ has equation

$$f_x(0, 0, 0)x + f_y(0, 0, 0)y + f_z(0, 0, 0)z + 1 = 0.$$

In Problems 72–74, give an example of:

72. A surface $z = f(x, y)$ such that the vector $\vec{i} - 2\vec{j} - \vec{k}$ is normal to the tangent plane at the point where $(x, y) = (0, 0)$.

73. A function $f(x, y, z)$ such that grad $f = 2\vec{i} + 3\vec{j} + 4\vec{k}$.

74. Two nonparallel unit vectors $\vec{u}$ and $\vec{v}$ such that $f_{\vec{u}}(0, 0, 0) = f_{\vec{v}}(0, 0, 0) = 0$, where $f(x, y, z) = 2x - 3y$.

Are the statements in Problems 75–78 true or false? Give reasons for your answer.

75. An equation for the tangent plane to the surface $z = x^2 + y^3$ at $(1, 1)$ is $z = 2 + 2x(x - 1) + 3y^2(y - 1)$.

76. There is a function $f(x, y)$ which has a tangent plane with equation $z = 0$ at a point (a, b).

77. There is a function with $\|$ grad $f\| = 4$ and $f_{\vec{k}} = 5$ at some point.

78. There is a function with $\|$ grad $f\| = 5$ and $f_{\vec{k}} = -3$ at some point.

79. Let $f(x, y, z)$ represent the temperature in °C at the point (x, y, z) with x, y, z in meters. Let $\vec{v}$ be your velocity in meters per second. Give units and an interpretation of each of the following quantities.

(a) $\|$ grad $f\|$ **(b)** grad $f \cdot \vec{v}$ **(c)** $\|$ grad $f\| \cdot \|\vec{v}\|$

14.6 THE CHAIN RULE

Composition of Functions of Many Variables and Rates of Change

The chain rule enables us to differentiate *composite functions*. If we have a function of two variables $z = f(x, y)$ and we substitute $x = g(t), y = h(t)$ into $z = f(x, y)$, then we have a composite function in which z is a function of t:

$$z = f(g(t), h(t)).$$

If, on the other hand, we substitute $x = g(u, v), y = h(u, v)$, then we have a different composite function in which z is a function of u and v:

$$z = f(g(u, v), h(u, v)).$$

The next example shows how to calculate the rate of change of a composite function.

Example 1 Corn production, C, depends on annual rainfall, R, and average temperature, T, so $C = f(R, T)$. Global warming predicts that both rainfall and temperature depend on time. Suppose that according to a particular model of global warming, rainfall is decreasing at 0.2 cm per year and temperature is increasing at 0.1°C per year. Use the fact that at current levels of production, $f_R = 3.3$ and $f_T = -5$ to estimate the current rate of change, dC/dt.