Exercises and Problems for Section 14.4

Exercises

In Exercises 1–14, find the gradient of the function. Assume the variables are restricted to a domain on which the function is defined.

1. $f(x, y) = \frac{3}{2}x^5 - \frac{4}{7}y^6$

2. $Q = 50K + 100L$

3. $f(m, n) = m^2 + n^2$

4. $z = xe^y$

5. $f(\alpha, \beta) = \sqrt{5\alpha^2 + \beta}$

6. $f(r, h) = \pi r^2 h$

7. $z = (x + y)e^y$

8. $f(K, L) = K^{0.3}L^{0.7}$

9. $f(r, \theta) = r\sin\theta$

10. $f(x, y) = \ln(x^2+y^2)$

11. $z = \sin(x/y)$

12. $z = \tan^{-1}(x/y)$

13. $f(\alpha, \beta) = \dfrac{2\alpha + 3\beta}{2\alpha - 3\beta}$

14. $z = x\dfrac{e^y}{x + y}$

In Exercises 15–22, find the gradient at the point.

15. $f(x, y) = x^2y + 7xy^3$, at $(1, 2)$

16. $f(m, n) = 5m^2 + 3n^4$, at $(5, 2)$

17. $f(r, h) = 2\pi rh + \pi r^2$, at $(2, 3)$

18. $f(x, y) = e^{\sin y}$, at $(0, \pi)$

19. $f(x, y) = \sin(x^2) + \cos y$, at $(\frac{\sqrt{\pi}}{2}, 0)$

20. $f(x, y) = \ln(x^2 + xy)$, at $(4, 1)$

21. $f(x, y) = 1/(x^2 + y^2)$, at $(-1, 3)$

22. $f(x, y) = \sqrt{\tan x + y}$, at $(0, 1)$

In Exercises 23–26, find the directional derivative $f_{\vec{u}}(1, 2)$ for the function f with $\vec{u} = (3\vec{i} - 4\vec{j})/5$.

23. $f(x, y) = xy + y^3$

24. $f(x, y) = 3x - 4y$

25. $f(x, y) = x^2 - y^2$

26. $f(x, y) = \sin(2x - y)$

27. If $f(x, y) = x^2y$ and $\vec{v} = 4\vec{i} - 3\vec{j}$, find the directional derivative at the point $(2, 6)$ in the direction of $\vec{v}$.

In Exercises 28–29, find the differential df from the gradient.

28. $\text{grad}\, f = y\vec{i} + x\vec{j}$

29. $\text{grad}\, f = (2x + 3e^y)\vec{i} + 3xe^y\vec{j}$

In Exercises 30–31, find grad f from the differential.

30. $df = 2xdx + 10ydy$

31. $df = (x + 1)ye^x dx + xe^x dy$

In Exercises 32–37, use the contour diagram of f in Figure 14.34 to decide if the specified directional derivative is positive, negative, or approximately zero.

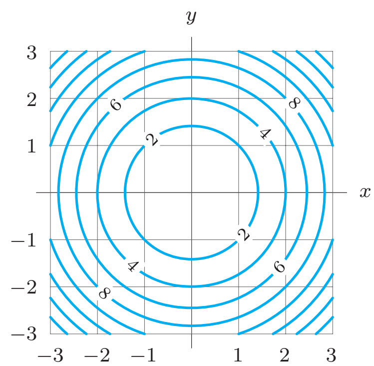

Figure 14.34

32. At point $(-2, 2)$, in direction $\vec{i}$.

33. At point $(0, -2)$, in direction $\vec{j}$.

34. At point $(0, -2)$, in direction $\vec{i} + 2\vec{j}$.

35. At point $(0, -2)$, in direction $\vec{i} - 2\vec{j}$.

36. At point $(-1, 1)$, in direction $\vec{i} + \vec{j}$.

37. At point $(-1, 1)$, in direction $-\vec{i} + \vec{j}$.

In Exercises 38–45, use the contour diagram of f in Figure 14.34 to find the approximate direction of the gradient vector at the given point.

38. $(-2, 0)$ **39.** $(0, -2)$ **40.** $(2, 0)$ **41.** $(0, 2)$

42. $(-2, 2)$ **43.** $(-2, -2)$ **44.** $(2, 2)$ **45.** $(2, -2)$

Problems

46. Let $f(P) = 15$ and $f(Q) = 20$ where $P = (3, 4)$ and $Q = (3.03, 3.96)$. Approximate the directional derivative of f at P in the direction of Q.

47. **(a)** Give Q, the point at a distance of 0.1 from $P = (4, 5)$ in the direction of $\vec{v} = -\vec{i} + 3\vec{j}$. Give five decimal places in your answer.

(b) Use P and Q to approximate the directional derivative of $f(x, y) = \sqrt{x + y}$ in the direction of $\vec{v}$.

(c) Give the exact value for the directional derivative you estimated in part (b).

48. Find the directional derivative of $f(x, y) = e^x \tan(y) + 2x^2y$ at the point $(0, \pi/4)$ in the following directions

(a) $\vec{i} - \vec{j}$ **(b)** $\vec{i} + \sqrt{3}\vec{j}$

49. Find the rate of change of $f(x, y) = x^2 + y^2$ at the point $(1, 2)$ in the direction of the vector $\vec{u} = 0.6\vec{i} + 0.8\vec{j}$.

50. **(a)** Let $f(x, y) = (x+y)/(1+x^2)$. Find the directional derivative of f at $P = (1, -2)$ in the direction of:

(i) $\vec{v} = 3\vec{i} - 2\vec{j}$ (ii) $\vec{v} = -\vec{i} + 4\vec{j}$

(b) What is the direction of greatest increase of f at P?

51. Let $f(x, y) = x^2y^3$. At the point $(-1, 2)$, find a vector

(a) In the direction of maximum rate of change.
(b) In the direction of minimum rate of change.
(c) In a direction in which the rate of change is zero.

52. You are at the point $(\pi/4, 1)$ and start to move in the direction of the point $(1 + \pi/4, 2)$. At what rate does the value of $f(x, y) = \sin(xy)$ change as you leave $(\pi/4, 1)$? Give your answer in units of f per unit distance.

53. **(a)** Let $f(x, y) = x^2 + \ln y$. Find the average rate of change of f as you go from $(3, 1)$ to $(1, 2)$.

(b) Find the instantaneous rate of change of f as you leave the point $(3, 1)$ heading toward $(1, 2)$.

54. **(a)** What is the rate of change of $f(x, y) = 3xy + y^2$ at the point $(2, 3)$ in the direction $\vec{v} = 3\vec{i} - \vec{j}$?

(b) What is the direction of maximum rate of change of f at $(2, 3)$?

(c) What is the maximum rate of change?

55. A student was asked to find the directional derivative of $f(x, y) = x^2e^y$ at the point $(1, 0)$ in the direction of $\vec{v} = 4\vec{i} + 3\vec{j}$. The student's answer was

$$f_{\vec{u}}(1, 0) = \text{grad}\, f(1, 0) \cdot \vec{u} = \frac{8}{5}\vec{i} + \frac{3}{5}\vec{j}.$$

(a) At a glance, how do you know this is wrong?
(b) What is the correct answer?

For Problems 56–60 use Figure 14.35, showing level curves of $f(x, y)$, to estimate the directional derivatives.

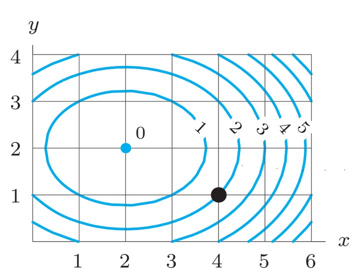

Figure 14.35

56. $f_{\vec{i}}(4, 1)$

57. $f_{\vec{j}}(4, 1)$

58. $f_{\vec{u}}(4, 1)$ where $\vec{u} = (\vec{i} - \vec{j})/\sqrt{2}$

59. $f_{\vec{u}}(4, 1)$ where $\vec{u} = (-\vec{i} + \vec{j})/\sqrt{2}$

60. $f_{\vec{u}}(4, 1)$ with $\vec{u} = (-2\vec{i} + \vec{j})/\sqrt{5}$

In Problems 61–64, check that the point $(2, 3)$ lies on the curve. Then, viewing the curve as a contour of $f(x, y)$, use grad $f(2, 3)$ to find a vector normal to the curve at $(2, 3)$ and an equation for the tangent line to the curve at $(2, 3)$.

61. $x^2 + y^2 = 13$

62. $xy = 6$

63. $y = x^2 - 1$

64. $(y - x)^2 + 2 = xy - 3$

65. The surface $z = g(x, y)$ is in Figure 14.36. What is the sign of each of the following directional derivatives?

(a) $g_{\vec{u}}(2, 5)$ where $\vec{u} = (\vec{i} - \vec{j})/\sqrt{2}$.
(b) $g_{\vec{u}}(2, 5)$ where $\vec{u} = (\vec{i} + \vec{j})/\sqrt{2}$.

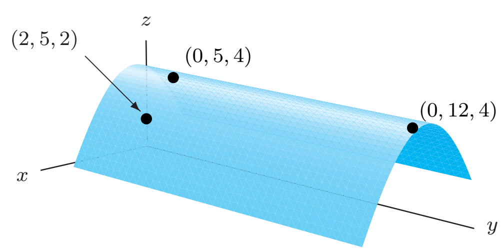

Figure 14.36

66. The table gives values of a differentiable function $f(x, y)$. At the point $(1.2, 0)$, into which quadrant does the gradient vector of f point? Justify your answer.

$x \backslash y$	-1	0	1
1.0	0.7	0.1	-0.5
1.2	4.8	4.2	3.6
1.4	8.9	8.3	7.7

67. Figure 14.37 represents the level curves $f(x, y) = c$; the values of f on each curve are marked. In each of the following parts, decide whether the given quantity is positive, negative or zero. Explain your answer.

(a) The value of $\nabla f \cdot \vec{i}$ at P.
(b) The value of $\nabla f \cdot \vec{j}$ at P.
(c) $\partial f/\partial x$ at Q.
(d) $\partial f/\partial y$ at Q.

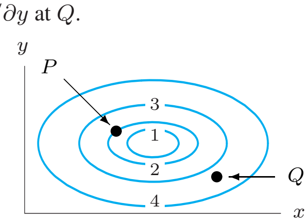

Figure 14.37

68. In Figure 14.37, which is larger: $\|\nabla f\|$ at P or $\|\nabla f\|$ at Q? Explain how you know.

In Problems 69–72, do the level curves of $f(x, y)$ cross the level curves of $g(x, y)$ at right angles? Sketch contour diagrams.

69. $f(x, y) = x + y$, $g(x, y) = x - y$

70. $f(x, y) = 2x + 3y$, $g(x, y) = 2x - 3y$

71. $f(x, y) = x^2 - y$, $g(x, y) = 2y + \ln |x|$

72. $f(x, y) = x^2 - y^2$, $g(x, y) = xy$

73. **(a)** Sketch the surface $z = f(x, y) = y^2$ in three dimensions.
(b) Sketch the level curves of f in the xy-plane.
(c) If you are standing on the surface $z = y^2$ at the point $(2, 3, 9)$, in which direction should you move to climb the fastest? (Give your answer as a 2-vector.)

74. You are standing above the point $(1, 3)$ on the surface $z = 20 - (2x^2 + y^2)$.

(a) In which direction should you walk to descend fastest? (Give your answer as a 2-vector.)
(b) If you start to move in this direction, what is the slope of your path?

75. Let P be a fixed point in the plane and let $f(x, y)$ be the distance from P to (x, y). Answer the following questions using geometric interpretations, not formulas.

(a) What are the level curves of f?
(b) In what direction does grad $f(x, y)$ point?
(c) What is the magnitude $\|$ grad $f(x, y)\|$?

76. The directional derivative of $z = f(x, y)$ at $(2, 1)$ in the direction toward the point $(1, 3)$ is $-2/\sqrt{5}$, and the directional derivative in the direction toward the point $(5, 5)$ is 1. Compute $\partial z/\partial x$ and $\partial z/\partial y$ at $(2, 1)$.

77. Consider the function $f(x, y)$. If you start at the point $(4, 5)$ and move toward the point $(5, 6)$, the directional derivative is 2. Starting at the point $(4, 5)$ and moving toward the point $(6, 6)$ gives a directional derivative of 3. Find ∇f at the point $(4, 5)$.

78. **(a)** For $g(x, y) = \sqrt{x^2 + 3y + 3}$, find grad $g(1, 4)$.
(b) Find the best linear approximation of $g(x, y)$ for (x, y) near $(1, 4)$.
(c) Use the approximation in part (b) to estimate $g(1.01, 3.98)$.

79. Find the directional derivative of $z = x^2 - y^2$ at the point $(3, -1)$ in the direction making an angle $\theta = \pi/4$ with the x-axis. In which direction is the directional derivative the largest?

80. The temperature H in °Fahrenheit y miles north of the Canadian border t hours after midnight is given by $H = 30 - 0.05y - 5t$. A moose runs north at a speed of 20 mph. At what rate does the moose perceive the temperature to be changing?

81. At a certain point on a heated plate, the greatest rate of temperature increase, 5° C per meter, is toward the northeast. If an object at this point moves directly north, at what rate is the temperature increasing?

82. You are climbing a mountain by the steepest route at a slope of 20° when you come upon a trail branching off at a 30° angle from yours. What is the angle of ascent of the branch trail?

83. Figure 14.39 is a graph of the directional derivative, $f_{\vec u}$, at the point (a, b) versus θ, the angle in Figure 14.38.

(a) Which points on the graph in Figure 14.39 correspond to the greatest rate of increase of f? The greatest rate of decrease?
(b) Mark points on the circle in Figure 14.38 corresponding to the points P, Q, R, S.
(c) What is the amplitude of the function graphed in Figure 14.39? What is its formula?

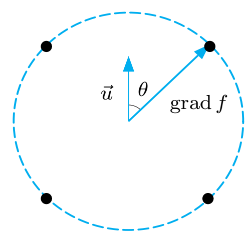

Figure 14.38

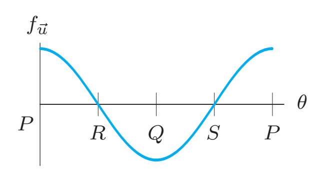

Figure 14.39

84. You are standing at the point $(1, 1, 3)$ on the hill whose equation is given by $z = 5y - x^2 - y^2$.

(a) If you choose to climb in the direction of steepest ascent, what is your initial rate of ascent relative to the horizontal distance?
(b) If you decide to go straight northwest, will you be ascending or descending? At what rate?
(c) If you decide to maintain your altitude, in what directions can you go?

85. In this problem we see another way of obtaining the formula $f_{\vec u}(a, b) = \text{grad } f(a, b) \cdot \vec u$. Imagine zooming in on a function $f(x, y)$ at a point (a, b). By local linearity, the contours around (a, b) look like the contours of a linear function. See Figure 14.40. Suppose you want to find the directional derivative $f_{\vec u}(a, b)$ in the direction of a unit vector $\vec u$. If you move from P to Q, a small distance h in the direction of $\vec u$, then the directional derivative is approximated by the difference quotient

$$\frac{\text{Change in } f \text{ between } P \text{ and } Q}{h}.$$

(a) Use the gradient to show that

$$\text{Change in } f \approx \|\text{grad } f\|(h\cos\theta).$$

(b) Use part (a) to obtain $f_{\vec u}(a, b) = \text{grad } f(a, b) \cdot \vec u$.

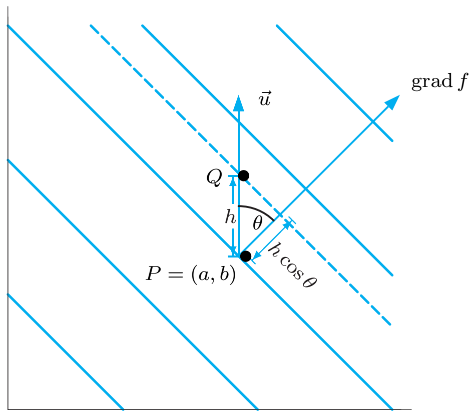

Figure 14.40

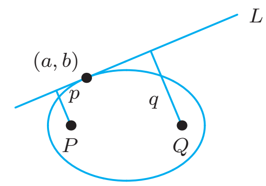

Figure 14.41

86. Let L be a line tangent to the ellipse $x^2/2 + y^2 = 1$ at the point (a, b). See Figure 14.41.

(a) Find a vector perpendicular to L.
(b) Find the distance p from $P = (-1, 0)$ to L as a function of a.
(c) Find the distance q from $Q = (1, 0)$ to L as a function of a.
(d) Show that $pq = 1$.

87. Let C be the contour C of $f(x, y)$ through (a, b) and grad $f(a, b) \neq \vec{0}$. Show that

(a) The vector $-f_y(a, b)\vec{i} + f_x(a, b)\vec{j}$ is tangent to C at (a, b).
(b) The slope of the line tangent to C at the point (a, b) is $-f_x(a, b)/f_y(a, b)$ if the tangent line is not vertical.

88. Let $\|\text{grad}\, f(x, y)\| = \|\text{grad}\, g(x, y)\|$ at a point P where these gradients are not the zero vector. Show that at P, the direction of the most rapid increase of $f + g$

(a) Increases f and g at equal rates.
(b) Bisects the angle between the contours of f and g that pass through P.

Strengthen Your Understanding

In Problems 89–91, explain what is wrong with the statement.

89. A function f has a directional derivative given by $f_{\vec{u}}(0, 0) = 3\vec{i} + 4\vec{j}$.

90. A function f has gradient grad $f(0, 0) = 7$.

91. The gradient vector grad $f(x, y)$ is perpendicular to the contours of f, and the closer together the contours for equally spaced values of f, the shorter the gradient vector.

In Problems 92–93, give an example of:

92. A unit vector $\vec{u}$ such that $f_{\vec{u}}(0, 0) < 0$, given that $f_x(0, 0) = 2$ and $f_y(0, 0) = 3$.

93. A contour diagram of a function with two points in the domain where the gradients are parallel but different lengths.

94. For the gradient $\nabla f(P)$ of f at a point P, describe the geometric interpretation of its

(a) Direction
(b) Magnitude
(c) Dot product with a unit vector $\vec{u}$

Are the statements in Problems 95–106 true or false? Give reasons for your answer.

95. If the point (a, b) is on the contour $f(x, y) = k$, then the slope of the line tangent to this contour at (a, b) is $f_y(a, b)/f_x(a, b)$.

96. The gradient vector grad $f(a, b)$ is a vector in 3-space.

97. $\text{grad}(fg) = (\text{grad}\, f) \cdot (\text{grad}\, g)$

98. The gradient vector grad $f(a, b)$ is tangent to the contour of f at (a, b).

99. If you know the gradient vector of f at (a, b) then you can find the directional derivative $f_{\vec{u}}(a, b)$ for any unit vector $\vec{u}$.

100. If you know the directional derivative $f_{\vec{u}}(a, b)$ for all unit vectors $\vec{u}$ then you can find the gradient vector of f at (a, b).

101. The directional derivative $f_{\vec{u}}(a, b)$ is parallel to $\vec{u}$.

102. The gradient grad $f(3, 4)$ is perpendicular to the vector $3\vec{i} + 4\vec{j}$.

103. If grad $f(1, 2) = \vec{i}$, then f decreases in the $-\vec{i}$ direction at $(1, 2)$.

104. If grad $f(1, 2) = \vec{i}$, then $f(10, 2) > f(1, 2)$.

105. At the point $(3, 0)$, the function $g(x, y) = x^2 + y^2$ has the same maximal rate of increase as that of the function $h(x, y) = 2xy$.

106. If $f(x,y) = e^{x+y}$, then the directional derivative in any direction $\vec{u}$ (with $\|\vec{u}\| = 1$) at the point $(0,0)$ is always less than or equal to $\sqrt{2}$.

Assume that $f(x,y)$ is a differentiable function. Are the statements in Problems 107–111 true or false? Explain your answer.

107. $f_{\vec{u}}(x_0, y_0)$ is a scalar.

108. $f_{\vec{u}}(a,b) = \|\nabla f(a,b)\|$

109. If $\vec{u}$ is tangent to the level curve of f at some point, then $\operatorname{grad} f \cdot \vec{u} = 0$ there.

110. There is always a direction in which the rate of change of f at (a,b) is 0.

111. There is a function with a point in its domain where $\|\operatorname{grad} f\| = 0$ and where there is a nonzero directional derivative.

14.5 GRADIENTS AND DIRECTIONAL DERIVATIVES IN SPACE

Directional Derivatives of Functions of Three Variables

We calculate directional derivatives of a function of three variables in the same way as for a function of two variables. If the function f is differentiable at the point (a,b,c), then the rate of change of $f(x,y,z)$ at the point (a,b,c) in the direction of a unit vector $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$ is

$$f_{\vec{u}}(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3.$$

This can be justified using local linearity in the same way as for functions of two variables.

Example 1 Find the directional derivative of $f(x,y,z) = xy + z$ at the point $(-1,0,1)$ in the direction of the vector $\vec{v} = 2\vec{i} + \vec{k}$.

Solution The magnitude of $\vec{v}$ is $\|\vec{v}\| = \sqrt{2^2+1} = \sqrt{5}$, so a unit vector in the same direction as $\vec{v}$ is

$$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{2}{\sqrt{5}}\vec{i} + 0\vec{j} + \frac{1}{\sqrt{5}}\vec{k}.$$

The partial derivatives of f are $f_x(x,y,z) = y$ and $f_y(x,y,z) = x$ and $f_z(x,y,z) = 1$. Thus,

$$\begin{aligned} f_{\vec{u}}(-1,0,1) &= f_x(-1,0,1)u_1 + f_y(-1,0,1)u_2 + f_z(-1,0,1)u_3 \\ &= (0)\left(\frac{2}{\sqrt{5}}\right) + (-1)(0) + (1)\left(\frac{1}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}}. \end{aligned}$$

The Gradient Vector of a Function of Three Variables

The gradient of a function of three variables is defined in the same way as for two variables:

$$\operatorname{grad} f(a,b,c) = f_x(a,b,c)\vec{i} + f_y(a,b,c)\vec{j} + f_z(a,b,c)\vec{k}.$$

Directional derivatives are related to gradients in the same way as for functions of two variables:

$$f_{\vec{u}}(a,b,c) = f_x(a,b,c)u_1 + f_y(a,b,c)u_2 + f_z(a,b,c)u_3 = \operatorname{grad} f(a,b,c)\cdot\vec{u}.$$

Since $\operatorname{grad} f(a,b,c)\cdot\vec{u} = \|\operatorname{grad} f(a,b,c)\|\cos\theta$, where θ is the angle between $\operatorname{grad} f(a,b,c)$ and $\vec{u}$, the value of $f_{\vec{u}}(a,b,c)$ is largest when $\theta = 0$, that is, when $\vec{u}$ is in the same direction as $\operatorname{grad} f(a,b,c)$. In addition, $f_{\vec{u}}(a,b,c) = 0$ when $\theta = \pi/2$, so $\operatorname{grad} f(a,b,c)$ is perpendicular to the level surface of f. The properties of gradients in space are similar to those in the plane: